On the stability analysis of parameterized set-valued inclusions.

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Overview

Parameterized generalized equations in their standard format

$$(PGE) \qquad\qquad \mathbf{0} \in f(p,x) + G(p,x)$$

where the problem data are:
$f : P \times \mathbb{X} \longrightarrow \mathbb{Y}$ single-valued mapping
$G : P \times \mathbb{X} \rightrightarrows \mathbb{Y}$ set-valued mapping
(P,d) is a metric (parameter) space
$(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|)$ are Banach spaces, $\mathbf{0}$ is the null vector in $\mathbb{Y}$.

Main historical motivations:
- equality/inequality constraint systems;
- cone constraint systems;
- optimality conditions;
- variational inequalities;
- fixed points;
- equilibria.

Main problem

The main problem addressed in this talk is to conduct a solution stability analysis for parameterized set-valued inclusions, namely for the following problem:

$$(PSV) \qquad F(p,x) \subseteq C,$$

where the problem data are:
$F : P \times \mathbb{X} \rightrightarrows \mathbb{Y}$ a set-valued mapping
$C \subseteq \mathbb{Y}$ a closed, convex cone, with $\{\mathbf{0}\} \neq C \neq \mathbb{Y}$.
$(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ are Banach spaces, $\mathbf{0}$ is the null vector in $\mathbb{Y}$.
The solution mapping $\mathcal{S} : P \rightrightarrows \mathbb{X}$ associated with (PSV) is defined as

$$\mathcal{S}(p) = \{x \in X : \ F(p,x) \subseteq C\}.$$

Key feature: (PGE) and (PSV) seem to be problems with a different nature. (PSV) can not be easily cast in the traditional generalized equation format.

A robust approach to uncertain optimization

Consider a cone constrained optimization problem with uncertainty

$$\min \varphi(x,\omega) \quad \text{subject to} \quad f(x,\omega) \in C,$$

where $x \in X$ is the decision vector, $\omega \in \Omega$ is a data element
$\varphi : \mathbb{X} \times \Omega \longrightarrow \mathbb{R}$ is the objective function (affected by uncertainty)
$f : \mathbb{X} \times \Omega \longrightarrow \mathbb{Y}$ and $C \subset \mathbb{Y}$ define the cone (uncertain) constraint.
If the decision environment is characterized by:

- a crude knowledge of the data: all is known about ω is that $\omega \in \Omega$;
- the constraint must be satisfied, whatever the actual realization of $\omega \in \Omega$ is;

then, after [BenNem98], a robust approach to uncertain optimization leads to consider $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ given by

$$F(x) = f(x,\Omega) = \{y \in \mathbb{Y} : \ f(x,\omega), \ \omega \in \Omega\},$$

and hence to face the set-valued inclusion $F(x) \subseteq C$.

Ideal efficient solutions in vector optimization

Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a mapping taking values in a space $(\mathbb{Y}, \leq_C)$ partially ordered by a cone C and let $R \subset \mathbb{X}$. $\bar{x} \in R$ is said to be an ideal C-efficient solution to the problem

$$C\text{-}\min f(x) \quad \text{subject to} \quad x \in R$$

if

$$f(\bar{x}) \leq_C f(x), \ \forall x \in R,$$

or equivalently, if

$$f(R) \subseteq f(\bar{x}) + C.$$

By setting $F_{f,R}(x) = f(R) - f(x)$, one readily sees that $\bar{x}$ is ideal C-efficient iff it solves the set-valued inclusion

$$F_{f,R}(x) \subseteq C.$$

Solution stability issues

Issues to be considered for a solution stability analysis: let $\mathcal{S} : P \rightrightarrows \mathbb{X}$ be the solution mapping associated with a (PSV).

1. (Solution existence: Fix $\bar{p} \in P$. Upon which conditions $\mathcal{S}(\bar{p}) \neq \varnothing$?)
2. Local solvability: If $\mathcal{S}(\bar{p}) \neq \varnothing$, upon which conditions $\mathcal{S}(p) \neq \varnothing$ for p near $\bar{p}$?
3. Local stability: If $\bar{p} \in \mathrm{int}\,\mathrm{dom}\,\mathcal{S}$ (i.e. $\mathcal{S}(p) \neq \varnothing,\ \forall p \in B(\bar{p}, \delta)$), when are the values $\mathcal{S}(p)$ 'near' $\mathcal{S}(\bar{p})$? In which sense 'near'? Can we measure such a nearness phenomenon?
4. Sensitivity: Whenever it is possible to quantify the changes of $\mathcal{S}$, under which conditions they are proportional to the changes of p. Can we measure the 'rate of change'? Can we provide first-order approximations of $\mathcal{S}$?

To embed the solution stability analysis of (PSV) in the framework of variational analysis one needs the above properties: let $\Phi : X \rightrightarrows Y$ be a mapping between metric spaces and $(\bar{x}, \bar{y}) \in \mathrm{gph}\, \Phi$. Φ is said:

- to be Lipschitz lower semicontinuous at $(\bar{x}, \bar{y})$ if $\exists \delta,\ \ell > 0$:

  $$(*) \qquad \Phi(x) \cap B(\bar{y}, \ell d(x, \bar{x})) \neq \varnothing, \ \forall x \in B(\bar{x}, \delta);$$

  the modulus of Lipschitz lower semicontinuity of Φ is defined as

  $$\mathrm{Liplsc}\, \Phi(\bar{x}, \bar{y}) = \inf\{\ell > 0 : \ \exists \delta > 0 \text{ for which } (*) \text{ holds}\}.$$

- to be calm at $(\bar{x}, \bar{y})$ if $\exists \delta,\ \zeta,\ \ell > 0$:

  $$(**) \qquad \Phi(x) \cap B(\bar{y}, \zeta) \subseteq B(\Phi(\bar{x}), \ell d(x, \bar{x})), \ \forall x \in B(\bar{x}, \delta);$$

  the modulus of calmness of Φ at $(\bar{x}, \bar{y})$ is defined as

  $$\mathrm{clm}\, \Phi(\bar{x}, \bar{y}) = \inf\{\ell > 0 : \ \exists \delta,\ \zeta > 0 \text{ for which } (**) \text{ holds}\}.$$

- to be Lipschitz upper semicontinuous at $\bar{x} \in X$ if $\exists \delta, \ell > 0$:

$$(***) \qquad \Phi(x) \subseteq B(\Phi(\bar{x}), \ell d(x, \bar{x})), \ \forall x \in B(\bar{x}, \delta);$$

  the modulus of Lipschitz upper semicontinuity at $\bar{x}$ is defined as

$$\mathrm{Lipusc}\, \Phi(\bar{x}) = \inf\{\ell > 0 : \ \exists \delta > 0 \text{ for which } (***) \text{ holds}\}.$$

- to have the Aubin property at $(\bar{x}, \bar{y})$ if $\exists \delta, \zeta, \ell > 0$:

$$(+) \quad \Phi(x_1) \cap B(\bar{y}, \zeta) \subseteq B(\Phi(x_2), \ell d(x_1, x_2)), \ \forall x_1, x_2 \in B(\bar{x}, \delta);$$

  the modulus of Aubin continuity of Φ at $(\bar{x}, \bar{y})$ is defined as

$$\mathrm{Lip}\, \Phi(\bar{x}, \bar{y}) = \inf\{\ell > 0 : \ \exists \delta, \zeta > 0 \text{ for which } (+) \text{ holds}\}.$$

- to be Lipschitz continuous around $\bar{x}$ if $\exists \delta, \ell > 0$:

$$(++) \qquad \mathrm{haus}(\Phi(x_1), \Phi(x_2)) \leq \ell d(x_1, x_2), \quad \forall x_1, x_2 \in B(\bar{x}, \delta).$$

Remark (Connections between Lipschitzian properties)

1. *Lipschitz lower semicontinuity and calmness are independent of each other.*
2. *Lipschitz upper semicontinuity* $\Rightarrow$ *Calmness* $\Leftarrow$ *Aubin property* $\Leftarrow$ *Lipschitz continuity*
3. *Whenever* $\Phi : X \longrightarrow Y$ *is single-valued near* $\bar{x}$*, Lipschitz lower semicontinuity* $=$ *Lipschitz upper semicontinuity* $=$ *'calmness' in the sense of Rockafellar, i.e.* $\exists\delta, \ell > 0$*:*

   $$d(\Phi(x), \Phi(\bar{x})) \le \ell d(x, \bar{x}), \ \forall x \in B(\bar{x}, \delta),$$

   which implies the classic continuity of Φ *at* $\bar{x}$*.*
4. Φ *has the Aubin property at* $(\bar{x}, \bar{y})$ *iff* Φ^{-1} *is metrically regular (equivalently, open at a linear rate) at* $(\bar{y}, \bar{x})$*.*

In order to conduct a study of the solution stability of (PSV), observe that

$$\mathcal{S}(p) = F^{+1}(p, \cdot)(C).$$

Standing assumptions:

1. $\mathrm{dom}\, F = P \times X$; F takes closed values;
2. (X, d) metrically complete.

Technique of analysis via merit function: define $\nu_{F,C} : P \times X \longrightarrow [0, +\infty]$ (it measures the inclusion violation)

$$\nu_{F,C}(p, x) = \mathrm{exc}(F(p, x), C) = \sup_{y \in F(p,x)} \mathrm{dist}\,(y, C),$$

where $\mathrm{dist}\,(y, C) = \inf_{c \in C} d(y, c)$.

Remark (Functional characterization of $\mathcal{S}$)

Under the above assumptions

$$\mathcal{S}(p) = F^{+1}(p, \cdot)(C) = [\nu_{F,C}(p, \cdot) = 0] = \nu_{F,C}(p, \cdot)^{-1}(0).$$

Basic tool for the analysis in metric spaces: slopes

Given $\varphi : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and $x_0 \in \varphi^{-1}(\mathbb{R})$ the extended-real value

$$|\nabla\varphi|(x_0) = \begin{cases} 0 & \text{if } x_0 \text{ is a local minimizer for } \varphi, \\ \displaystyle\limsup_{x \to x_0} \frac{\varphi(x_0) - \varphi(x)}{d(x, x_0)} & \text{otherwise,} \end{cases}$$

is called (strong) slope of φ at x_0. The extended-real value

$$\overline{|\nabla\varphi|}^{>}(x_0) = \liminf_{\substack{x \to x_0 \\ \varphi(x) \downarrow \varphi(x_0)}} |\nabla\varphi|(x).$$

is called strict outer slope of φ at x_0.
If $\nu : P \times X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and $(\bar{p}, \bar{x}) \in P \times X$, the extended-real value

$$|\nabla_x\nu|(\bar{p}, \bar{x}) = \begin{cases} 0 & \text{if } (\bar{p}, \bar{x}) \text{ is a local minimizer for } \nu, \\ \displaystyle\limsup_{x \to \bar{x}} \frac{\nu(\bar{p}, \bar{x}) - \nu(\bar{p}, x)}{d(x, \bar{x})} & \text{otherwise} \end{cases}$$

is called partial slope of ν, with respect to x, at $(\bar{p}, \bar{x})$.

A sufficient condition for Lipschitz lower semicontinuity of $\mathcal{S}$

Define the following ad hoc variant of partial strict outer slope

$$\overline{|\nabla_x \nu_{F,C}|}^{>}(\bar{p},\bar{x}) = \liminf_{\substack{(p,x)\to(\bar{p},\bar{x}) \\ \nu_{F,C}(p,x)\downarrow\nu_{F,C}(\bar{p},\bar{x})}} |\nabla \nu_{F,C}(p,\cdot)|(x).$$

Theorem (Lipschitz l.s.c. of $\mathcal{S}$)

With reference to a (PSV), let $\bar{p}\in P$ and $\bar{x}\in\mathcal{S}(\bar{p})$. Suppose:
(i) $\exists\delta>0$: $F(p,\cdot):X\rightrightarrows Y$ l.s.c. on X, $\forall p\in B(\bar{p},\delta)$;
(ii) $F(\cdot,\bar{x}):P\rightrightarrows Y$ Lipschitz u.s.c. at $\bar{p}$;
(iii) $\overline{|\nabla_x \nu_{F,C}|}^{>}(\bar{p},\bar{x})>0$.
Then, $\mathcal{S}:P\rightrightarrows X$ is Lipschitz l.s.c. at $(\bar{p},\bar{x})$ and

$$\mathrm{Liplsc}\,\mathcal{S}(\bar{p},\bar{x}) \le \frac{\mathrm{Lipusc}\,F(\cdot,\bar{x})(\bar{p})}{\overline{|\nabla_x \nu_{F,C}|}^{>}(\bar{p},\bar{x})}.$$

A sufficient condition for the calmness of $\mathcal{S}$

By using the following different variant of partial strict outer slope

$$\overline{|\nabla \nu_{F,C}(\bar{p},\cdot)|}^{>}(\bar{x}) = \liminf_{\substack{x \to \bar{x} \\ \nu_{F,C}(\bar{p},x) \downarrow \nu_{F,C}(\bar{p},\bar{x})}} |\nabla \nu_{F,C}(\bar{p},\cdot)|(x),$$

one obtains

Theorem (Calmness of $\mathcal{S}$)

With reference to a (PSV), let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose:
(i) $F(\bar{p},\cdot) : X \rightrightarrows Y$ l.s.c. on X;
(ii) F locally Lipschitz near $(\bar{p},\bar{x})$;
(iii) $\overline{|\nabla \nu_{F,C}(\bar{p},\cdot)|}^{>}(\bar{x}) > 0$.
Then, $\mathcal{S} : P \rightrightarrows X$ is calm at $(\bar{p},\bar{x})$ and

$$\operatorname{clm} \mathcal{S}(\bar{p},\bar{x}) \le \frac{\operatorname{Lip} F(\bar{p},\bar{x})}{\overline{|\nabla \nu_{F,C}(\bar{p},\cdot)|}^{>}(\bar{x})}.$$

A sufficient condition for Lipschitz upper semicontinuity of $\mathcal{S}$

Define modulus of Lipschitz continuity with respect to p, uniformly in x

$$\mathrm{Lip_p}\, F(\bar p, X) = \inf\{\ell > 0 : \exists \delta > 0 : \sup_{x\in X} \mathrm{haus}(F(p_1, x), F(p_2, x)) \le \ell d(p_1, p_2) \quad \forall p_1,\ p_2 \in \mathrm{B}(\bar p, \delta)\}$$

and a nonlocal regularization of the slope

$$\tau_{\bar p} = \inf\{|\nabla \nu_{F,C}(\bar p, \cdot)|(x) :\ x \in X \backslash \mathcal{S}(\bar p)\}$$

Theorem (Lipschitz upper semicontinuity of $\mathcal{S}$)

With reference to (PSV), let $\bar p \in P$, with $\mathcal{S}(\bar p) \ne \varnothing$. Suppose:
(i) $F(\bar p, \cdot) : X \rightrightarrows Y$ l.s.c. on X;
(ii) F is locally Lipschitz near $\bar p$ with respect to p, uniformly in $x \in X$;
(iii) it is $\tau_{\bar p} > 0$.
Then, the solution mapping $\mathcal{S} : P \rightrightarrows X$ is Lipschitz u.s.c. at $\bar p$ and

$$\mathrm{Lipusc}\, \mathcal{S}(\bar p) \le \frac{\mathrm{Lip_p}\, F(\bar p, X)}{\tau_{\bar p}}.$$

A sufficient condition for Aubin property of $\mathcal{S}$

Given $\delta > 0$, define
$\sigma_{\nabla}(\delta) = \inf\{|\nabla_x \nu_{F,C}|(p,x) : \ (p,x) \in [B(\bar{p},\delta) \times B(\bar{x},\delta)] \backslash \mathrm{gph}\, \mathcal{S}\}$.

Theorem (Local solvability and Aubin property of $\mathcal{S}$)

With reference to (PSV), let $\bar{p} \in P$, with $\mathcal{S}(\bar{p}) \neq \varnothing$. Suppose:
(i) $\exists \delta_1 > 0$: $F(p,\cdot)$ l.s.c. on X, $\forall p \in B(\bar{p},\delta_1)$;
(ii) $F(\cdot,\bar{x})$ Hausdorff C-u.s.c. at $\bar{p}$;
(iii) $\exists \delta_2 > 0$: $\sigma_{\nabla}(\delta_2) > 0$.
Then, $\exists \eta, \zeta > 0$ such that
(t) $\mathcal{S}(p) \cap B(\bar{x},\eta) \neq \varnothing$, $\forall p \in B(\bar{p},\zeta)$;
(tt) it holds $\mathrm{dist}\,(x,\mathcal{S}(p)) \leq \frac{\nu_{F,C}(p,x)}{\sigma_{\nabla}(\delta_2)}$, $\forall (p,x) \in B(\bar{p},\zeta) \times B(\bar{x},\eta)$.
Moreover, if
(v) $\exists \tau, s > 0$: $\forall x \in B(\bar{x},s)$ $F(\cdot,x)$ Lipschitz with rank ℓ in $B(\bar{p},\tau)$, then
(ttt) $\mathcal{S}$ has the Aubin property at $(\bar{p},\bar{x})$ and $\mathrm{Lip}\,\mathcal{S}(\bar{p},\bar{x}) \leq \ell/\sigma_{\nabla}(\delta_2)$.

Analysis in Banach spaces

In the more structured setting of Banach spaces, one can:

- provide verifiable estimates of the partial strict outer slope of $\nu_{F,C}$ in terms of problem data (F and C);
- study the first order behaviour of $\mathcal{S}$ by means of its generalized derivatives (in the spirit of implicit function theorems).

Standing assumption:

- C is a closed, convex nontrivial ($\{\mathbf{0}\} \neq C \neq \mathbb{Y}$) cone.

Analysis tools in Banach spaces

Definition (metric C-increase)

A set-valued mapping $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be
(i) metrically C-increasing on $\mathbb{X}$ if $\exists \alpha > 1$ such that

$$(*) \qquad \forall x \in \mathbb{X},\ \forall r > 0\ \exists u \in B(x, r) : B(\Phi(u), \alpha r) \subseteq B(\Phi(x) + C, r);$$

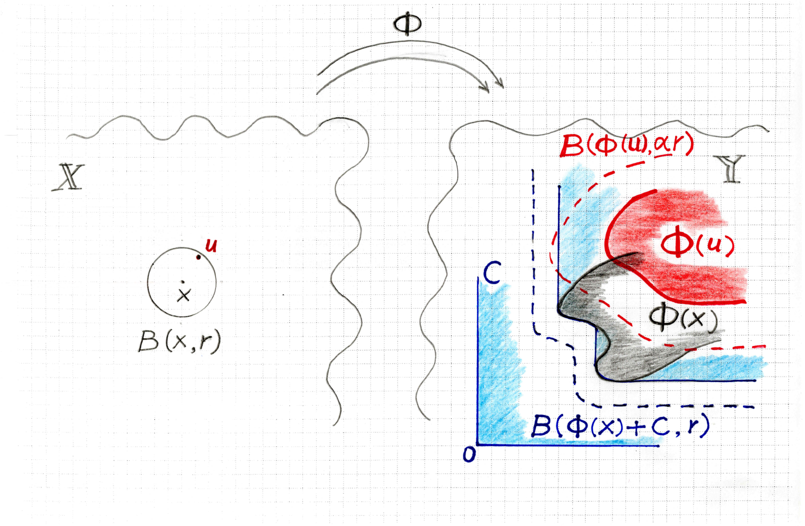
$$\mathrm{inc}\, \Phi = \sup\{\alpha > 1 : \ \text{inclusion } (*) \text{ holds}\}$$

is the exact bound of metric C-increase of Φ.
(ii) metrically C-increasing around $\bar{x} \in \mathrm{dom}\, \Phi$ if $\exists \delta > 0$, $\exists \alpha > 1$:

$$\forall x \in B(\bar{x}, \delta),\ \forall r \in (0, \delta)\ \exists u \in B(x, r) : \\ B(\Phi(u), \alpha r) \subseteq B(\Phi(x) + C, r); \qquad (**)$$

$$\mathrm{inc}\, \Phi(\bar{x}) = \sup\{\alpha > 1 : \ \text{inclusion } (**) \text{ holds}\}$$

is the exact bound of metric C-increase of Φ around $\bar{x}$.


Φ
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B(Φ(x)+C,r)
0

Remark (Scalar and single-valued case)

If $C = (-\infty, 0]$ and $\varphi : \mathbb{X} \longrightarrow \mathbb{R}$, $\Phi = \varphi$ is metrically C-increasing around $\bar{x}$ iff $\exists \mathrm{inc}\, \Phi(\bar{x}) > 1$ and $\delta > 0$ such that for any $\alpha \in (1, \mathrm{inc}\, \Phi(\bar{x}))$ and $r \in (0, \delta)$

$$\inf_{x \in B(\bar{x}, r)} \varphi(x) \leq \varphi(\bar{x}) - (\alpha - 1) r.$$

Recall that, according to the decrease principle (in the sense of Clarke-Ledyaev),

$$\exists r, c > 0 : \inf_{x \in B(\bar{x}, r)} |\nabla \varphi|(x) \geq c \quad \Rightarrow \quad \inf_{x \in B(\bar{x}, r)} \varphi(x) \leq \varphi(\bar{x}) - cr.$$

So, the metric C-increase property can be regarded as a set-valued counterpart of a decrease principle (nonstationariety condition).

Some example and connections with the regularity behaviour

Let $\Lambda : \mathbb{X} \to \mathbb{R}^m$ be a linear bounded operator, $C = \mathbb{R}^m_+$, with $m \geq 2$. Recall that Λ is onto iff $\exists \alpha > 0$: $\Lambda\mathbb{B} \supseteq \alpha\mathbb{B}$ (Banach open mapping principle). Define

$$\mathrm{sur}\,\Lambda = \sup\{\alpha > 0 : \ \Lambda\mathbb{B} \supseteq \alpha\mathbb{B}\}, \qquad \text{(Banach constant of } \Lambda\text{)}$$

then, Λ is onto iff $\mathrm{sur}\,\Lambda > 0$ and the following characterization holds

$$\mathrm{sur}\,\Lambda = \inf_{\|u^*\|=1} \|\Lambda^\top u^*\| = \mathrm{dist}\left(\mathbf{0}^*, \Lambda^\top \mathbb{S}^*\right),$$

where $\Lambda^\top$ denotes the adjoint operator to Λ and $\mathbb{S}^*$ the dual unit sphere.

Example (Linear epimorphism as a metrically C-increasing mapping)

Let $\Lambda : \mathbb{X} \to \mathbb{R}^m$ be such that $\mathrm{sur}\,\Lambda > m \geq 2$. Then, Λ is metrically $\mathbb{R}^m_+$-increasing on $\mathbb{X}$ and $\mathrm{inc}\,\Lambda \geq \sqrt{m}$.

Some example and connections with the regularity behaviour

Let $f : \mathbb{X} \to \mathbb{R}^m$ be a smooth mapping, $C = \mathbb{R}^m_+$, with $m \geq 2$. Recall that f covers at a linear rate around $\bar{x} \in \mathbb{X}$ if $\exists \alpha,\ \delta > 0$:

$$(*) \qquad f(B(x,r)) \supseteq B(f(x), \alpha r),\ \forall r \in (0,\delta), \quad \forall x \in\ B(\bar{x}\,\delta).$$

Define

$$\mathrm{sur}\,(f;\bar{x}) = \sup\{\alpha > 0 :\ \exists \delta > 0 :\ (*) \text{ holds}\}.$$

Letting $\widehat{\mathrm{D}}f(\bar{x})$ be the strict (Fréchet) derivative of f at $\bar{x}$, f covers at a linear rate around $\bar{x}$ iff (by the Lyusternik-Graves theorem)

$$0 < \mathrm{sur}\,(f;\bar{x}) = \inf_{\|u^*\|=1} \|\widehat{\mathrm{D}}f(\bar{x})^\top u^*\| = \mathrm{dist}\left(\mathbf{0}^*, \widehat{\mathrm{D}}f(\bar{x})^\top \mathbb{S}^*\right).$$

Example (Locally surjective/metrically regular mappings)

Let $f : \mathbb{X} \to \mathbb{R}^m$ be strictly differentiable at $\bar{x}$ and such that $\mathrm{sur}\,(f;\bar{x}) > m \geq 2$. Then, f is metrically $\mathbb{R}^m_+$-increasing around $\bar{x}$ and $\mathrm{inc}\, f(\bar{x}) \geq \sqrt{m}$.

Other tools in Banach spaces

Definition (C-concavity for set-valued mappings)

A set-valued mapping $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be C-concave on $\mathbb{X}$ if

$$\Phi(tx_1 + (1-t)x_2) \subseteq t\Phi(x_1) + (1-t)\Phi(x_2) + C, \ \forall x_1, x_2 \in \mathbb{X}, \ \forall t \in [0,1].$$

Example (Ioffe's fans)

A set-valued mapping $H : \mathbb{X} \rightrightarrows \mathbb{Y}$ is called fan if it fulfils the properties:
(i) $\mathbf{0} \in H(\mathbf{0})$;
(ii) H takes convex values;
(iii) $H(tx) = tH(x), \ \forall t > 0, \ \forall x \in \mathbb{X}$;
(iv) $H(x_1 + x_2) \subseteq H(x_1) + H(x_2), \ \forall x_1, x_2 \in \mathbb{X}$.
For instance, if $\mathcal{G}$ is a convex and weakly closed subset of linear bounded operators between $\mathbb{X}$ and $\mathbb{Y}$, then

$$\Phi_{\mathcal{G}}(x) = \{y \in \mathbb{Y} : \ y = \Lambda x, \ \Lambda \in \mathcal{G}\} \ \text{ is a fan.}$$

Estimating slopes by means of (convex) subdifferentials

Remark

If $F : P \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is C-concave with respect to x, for each $p \in P$, then $\nu_{F,C} : \mathbb{X} \longrightarrow [0, +\infty]$ is convex.

As a consequence $|\nabla_x \nu_{F,C}|$ can be estimated by $\mathrm{dist}\,(\mathbf{0}^*, \partial\nu_{F,C}(p,\cdot)(x))$. In turn $\mathrm{dist}\,(\mathbf{0}^*, \partial\nu_{F,C}(p,\cdot)(x))$ can be estimated from below by employing the C-increase property: under C-concavity and C-increase assumptions on Φ, one has $\exists\eta > 0$ such that

$$\inf_{u \in \mathbb{B}} \nu'_\Phi(x; u) \leq 1 - \mathrm{inc}\,\Phi(\bar{x}), \quad \forall x \in \ B(\bar{x}, \eta) \cap [\nu_\Phi > 0]$$

and

$$\mathrm{dist}\,(\mathbf{0}^*, \partial\nu_\Phi(p,\cdot)(x)) \geq \mathrm{inc}\,\Phi(\bar{x}) - 1, \quad \forall x \in \ B(\bar{x}, \eta) \cap [\nu_\Phi > 0].$$

As a consequence, one obtains the following estimates

Theorem (Slope estimates in the C-concave case)

With reference to (PSV), let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose $F(p, \cdot)$ l.s.c., C-concave and bounded-valued away from C, for every $p \in B(\bar{p}, \delta)$. If

(i) F is metrically C-increasing w.r.t. x, uniformly in p, around $(\bar{p}, \bar{x})$, then

$$\overline{|\nabla_x \nu_{F,C}|}^{>}(\bar{p}, \bar{x}) \geq \mathrm{inc}\, F_x(\bar{p}, \bar{x}) - 1 > 0;$$

(ii) $F(\bar{p}, \cdot)$ is metrically C-increasing around $\bar{x}$, then

$$\overline{|\nabla \nu_{F,C}(\bar{p}, \cdot)|}^{>}(\bar{x}) \geq \mathrm{inc}\, F(\bar{p}, \cdot)(\bar{x}) - 1 > 0;$$

(iii) $F(\bar{p}, \cdot)$ is metrically C-increasing on $\mathbb{X}$, then

$$\tau_{\bar{p}} \geq \mathrm{inc}\, F(\bar{p}, \cdot) - 1 > 0.$$

Differentiation notions for set-valued mappings

Let $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $x_0 \in \mathrm{dom}\,\Phi$. A positively homogeneous set-valued mapping $H_\Phi(x_0; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be

- an outer prederivative of Φ at x_0 if $\forall \epsilon > 0\ \exists \delta > 0$:

$$\Phi(x) \subseteq \Phi(x_0) + H_\Phi(x_0; x - x_0) + \epsilon \|x - x_0\| \mathbb{B}, \ \forall x \in B(x_0, \delta);$$

- an inner prederivative of Φ at x_0 if $\forall \epsilon > 0\ \exists \delta > 0$:

$$\Phi(x_0) + H_\Phi(x_0; x - x_0) \subseteq \Phi(x) + \epsilon \|x - x_0\| \mathbb{B}, \ \forall x \in B(x_0, \delta).$$

Remark (Connection with the Bouligand derivative)

Whenever Φ and $H_\Phi(x_0; \cdot)$ are single-valued around x_0, $H_\Phi(x_0; \cdot)$ turns out to be a Bouligand derivative of Φ at x_0 in the sense of Robinson.

The contingent derivative of Φ at $(x_0, y_0) \in \mathrm{gph}\,\Phi$ is the set-valued mapping $\mathrm{D}\Phi(x_0, y_0) : \mathbb{X} \rightrightarrows \mathbb{Y}$ defined via its graph by the equality

$$\mathrm{gph}\,\mathrm{D}\Phi(x_0, y_0) = \mathrm{T}(\mathrm{gph}\,\Phi; (x_0, y_0)),$$

where $\mathrm{T}(\mathrm{gph}\,\Phi; (x_0, y_0))$ denotes the contingent (Bouligand tangent) cone to $\mathrm{gph}\,\Phi$ at (x_0, y_0).

The Fréchet coderivative of Φ at $(x_0, y_0) \in \mathrm{gph}\,\Phi$ is the set-valued mapping $\widehat{\mathrm{D}}^*\Phi(x_0, y_0) : \mathbb{Y}^* \rightrightarrows \mathbb{X}^*$ defined by

$$\widehat{\mathrm{D}}^*\Phi(x_0, y_0)(y^*) = \{x^* \in \mathbb{X}^* : \ (x^*, -y^*) \in \widehat{\mathrm{N}}(\mathrm{gph}\,\Phi; (x_0, y_0))\}, \quad \text{where}$$

$$\widehat{\mathrm{N}}(\mathrm{gph}\,\Phi; (x_0, y_0)) = \left\{ w^* \in \mathbb{X}^* \times \mathbb{Y}^* : \limsup_{\mathrm{gph}\,\Phi \ni (x,y) \to (x_0, y_0)} \frac{\langle w^*, (x, y) - (x_0, y_0)\rangle}{\|(x, y) - (x_0, y_0)\|} \leq 0 \right\}$$

denotes the Fréchet normal cone to $\mathrm{gph}\,\Phi$ at (x_0, y_0).

Error bounds and Aubin property of $\mathcal{S}$

Given an outer prederivative $H_{F(p,\cdot)} : \mathbb{X} \rightrightarrows \mathbb{Y}$ of $F(p,\cdot)$ at x, define

$$\sigma_{H_{F(p,\cdot)}}(x) = \sup_{u \in \mathbb{S}} |C \overset{*}{-} H_{F(p,\cdot)}(x; u)|,$$

where

$$K \overset{*}{-} S = \{y \in \mathbb{Y} : \ y + S \subseteq K\} \qquad \text{(Pontryagin difference of } K \text{ and } S)$$

and

$$|K \overset{*}{-} S| = \sup\{r > 0 : \ r\mathbb{B} \subseteq K \overset{*}{-} S\} = \sup\{r > 0 : \ r\mathbb{B} + S \subseteq K\}$$

measures how much S is inner to K.

A sufficient condition for the Aubin property of $\mathcal{S}$

Let $(\bar{p}, \bar{x}) \in P \times \mathbb{X}$, $\delta > 0$ and
$\sigma_H(\bar{p}, \bar{x}; \delta) = \inf\{\sigma_{H_{F(p,\cdot)}}(x) : \ (p, x) \in [B(\bar{p}, \delta) \times B(\bar{x}, \delta]\backslash \mathrm{gph}\, \mathcal{S}\}$.

Theorem (Local solvability, error bound and Aubin property of $\mathcal{S}$)

With reference to (PSV), let $\bar{p} \in P$ and $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose:
(i) $\exists \delta_1 > 0 : \ \forall p \in B(\bar{p}, \delta_1)$, $F(p, \cdot)$ l.s.c. on $\mathbb{X}$;
(ii) $F(\cdot, \bar{x})$ Hausdorff C-u.s.c. at $\bar{p}$;
(iii) $\exists \delta_2 > 0 : \ \forall p \in B(\bar{p}, \delta_2)$ $F(p, \cdot)$ admits an outer prederivative $H_{F(p,\cdot)}(x; \cdot)$ at each $x \in B(\bar{x}, \delta_2)$ and $\sigma_H(\bar{p}, \bar{x}; \delta_2) > 0$.
Then, $\exists \eta, \zeta > 0$: $\mathcal{S}(p) \cap B(\bar{x}, \eta) \neq \varnothing$, $\forall p \in B(\bar{p}, \zeta)$ and
$\mathrm{dist}\,(x, \mathcal{S}(p)) \leq \nu_{F,C}(p, x)/\sigma_H(\bar{p}, \bar{x}; \delta)$, $\forall (p, x) \in B(\bar{p}, \zeta) \times B(\bar{x}, \eta)$.
(v) If, in addition, $\exists \tau, s > 0 : \ \forall x \in B(\bar{x}, s)$ $F(\cdot, x)$ is Lipschitz with constant ℓ in $B(\bar{p}, \tau)$, then $\mathcal{S}$ has the Aubin property at $(\bar{p}, \bar{x})$ and
$\mathrm{Lip}\, \mathcal{S}(\bar{p}, \bar{x}) \leq \frac{\ell}{\sigma_H(\bar{p}, \bar{x}; \delta_2)}$.

Henceforth, suppose also $(\mathbb{P}, \|\cdot\|)$ to be a normed linear space. In such a setting, it is possible to investigate geometric properties of $\mathcal{S}$.

A remarkable consequence on the first order (graphical) approximation of $\mathcal{S}$.

Corollary (Graphical derivative of $\mathcal{S}$)

Under hypotheses (i)-(v) of the previous theorem, $\mathrm{dom}\, \mathrm{D}\mathcal{S}(\bar{p}, \bar{x}) = \mathbb{P}$ *and* $\mathrm{D}\mathcal{S}(\bar{p}, \bar{x}) : \mathbb{P} \rightrightarrows \mathbb{X}$ *is Lipschitz continuous on* $\mathbb{P}$*, with constant*

$$\mathrm{Lip}\mathrm{D}\mathcal{S}(\bar{p}, \bar{x}) \leq \frac{\ell}{\sigma_H(\bar{p}, \bar{x}; \delta_2)}$$

Geometric properties of $\mathcal{S}$ under C-concavity of F

Proposition (Convexity of $\mathcal{S}$)

With reference to (PSV), suppose that $F : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is C-concave on $\mathbb{P} \times \mathbb{X}$. Then, $\mathcal{S} : \mathbb{P} \rightrightarrows \mathbb{X}$ is a convex set-valued mapping. If $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathcal{S}$, then $\mathrm{D}\mathcal{S}(\bar{p}, \bar{x}) : \mathbb{P} \rightrightarrows \mathbb{X}$ is a closed convex process.

By combining the Aubin property and the convexity of $\mathcal{S}$, one obtains

Corollary (Lipschitz continuity of $\mathcal{S}$ under truncation)

With reference to (PSV), let $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathcal{S}$. Suppose that all hypotheses (i)-(v) are satisfied and F is C-concave on $\mathbb{P} \times \mathbb{X}$. Then $\mathcal{S}$ has a Lipschitz continuous (not necessarily, single-valued) graphical localization around $(\bar{p}, \bar{x})$, i.e. $\exists V$ neighbourhood of $\bar{p}$ and $\exists U$ of $\bar{x}$ such that the set-valued mapping $p \rightsquigarrow \mathcal{S}(p) \cap U$ is Lipschitz continuous on V.

Approximations of $\mathrm{D}\mathcal{S}(\bar{p}, \bar{x})$

In the absence of convexity of $\mathcal{S}$, there are formulae which provide inner/outer approximations of $\mathrm{D}\mathcal{S}(\bar{p}, \bar{x})$ in terms of prederivatives of F.

Theorem (Inner approximation of $\mathrm{D}\mathcal{S}(\bar{p}, \bar{x})$)

With reference of (PSV), let $(\bar{p}, \bar{x}) \in \mathrm{gph}\, \mathcal{S}$. Suppose that:
(i) $F(\cdot, \bar{x})$ Hausdorff C-u.s.c. at $\bar{p}$;
(ii) $\exists \delta_1 > 0$: $\forall p \in B(\bar{p}, \delta_1)$, $F(p, \cdot)$ l.s.c. on $\mathbb{X}$;
(iii) $\exists \delta_2 > 0$: $\forall p \in B(\bar{p}, \delta_2)$, $F(p, \cdot)$ has $H_{F(p,\cdot)}(x; \cdot)$ as an outer prederivative at each $x \in B(\bar{x}, \delta_2)$ and $\sigma_H(\bar{p}, \bar{x}; \delta_2) > 0$;
(iv) F has $H_F((\bar{p}, \bar{x}); \cdot)$ as an outer (joint) prederivative at $(\bar{p}, \bar{x})$.
Then, it holds

$$\mathrm{D}\mathcal{S}(\bar{p}, \bar{x})(p) \supseteq H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(C), \quad \forall p \in \mathbb{P}.$$

Theorem (Outer approximation of $\mathrm{D}\mathcal{S}(\bar{p}, \bar{x})$)

With reference of (PSV), let $(\bar{p}, \bar{x}) \in \mathrm{gph}\, \mathcal{S}$. Suppose that:
(i) F has $H_F((\bar{p}, \bar{x}); \cdot)$ as an inner (joint) prederivative at $(\bar{p}, \bar{x})$;
(ii) $H_F((\bar{p}, \bar{x}); \cdot) : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is l.s.c. on $\mathbb{P} \times \mathbb{X}$.
Then, it holds

$$(\diamond) \qquad \mathrm{D}\mathcal{S}(\bar{p}, \bar{x})(p) \subseteq \bigcap_{y \in F(\bar{p}, \bar{x})} H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(\mathrm{T}(C; y)), \quad \forall p \in \mathbb{P}.$$

Remark (Case $F(\bar{p}, \bar{x}) \subseteq \mathrm{int}\, C$)

In the case $F(\bar{p}, \bar{x}) \subseteq \mathrm{int}\, C$, formula $(\diamond)$ gives no information. On the other hand, if F is u.s.c. at $(\bar{p}, \bar{x})$, in the present circumstance one has

$$\mathrm{D}\mathcal{S}(\bar{p}, \bar{x})(p) = \mathbb{X}, \quad \forall p \in \mathbb{P}.$$

A dual first-order approximation of $\mathcal{S}$

It is possible to provide further information about $\mathcal{S}$ near $(\bar{p}, \bar{x})$ via its Fréchet coderivative.

Theorem (Coderivative of $\mathcal{S}$)

With reference of (PSV), let $(\bar{p}, \bar{x}) \in \mathrm{gph}\, \mathcal{S}$. Suppose that:
(i) $F(\cdot, \bar{x})$ Hausdorff C-u.s.c. at $\bar{p}$;
(ii) $\exists \delta_1 > 0$: $\forall p \in B(\bar{p}, \delta_1)$, $F(p, \cdot)$ l.s.c. on $\mathbb{X}$;
(iii) $\exists \delta_2 > 0$: $\forall p \in B(\bar{p}, \delta_2)$, $F(p, \cdot)$ has $H_{F(p,\cdot)}(x; \cdot)$ as an outer prederivative at each $x \in B(\bar{x}, \delta_2)$ and $\sigma_H(\bar{p}, \bar{x}; \delta_2) > 0$.
Then $\exists \zeta, \eta > 0$: $\forall (p, x) \in [\mathrm{int}\, B(\bar{p}, \zeta) \times \mathrm{int}\, B(\bar{x}, \eta)] \cap \mathrm{gph}\, \mathcal{S}$ it holds

$$\widehat{\mathrm{D}}^* \mathcal{S}(p, x)(x^*) = \{p^* \in \mathbb{P}^* : \ (p^*, -x^*) \in \mathrm{cone}\, \widehat{\partial} \nu_{F,C}(p, x)\},$$

where $\widehat{\partial} \nu_{F,C}(p, x)$ denotes the Fréchet subdifferential of $\nu_{F,C}$ at (p, x) and cone the conic hull of a set.

Some open questions (representations of $\partial\nu_{F,C}$)

Let $\nu_{F,C} : \mathbb{X} \longrightarrow [0, +\infty]$ be defined by $\nu_{F,C}(x) = \sup_{y\in F(x)} \mathrm{dist}\,(y, C)$ and let $\overline{F} : \mathbb{X} \rightrightarrows \mathbb{Y}$ be defined by

$$\overline{F}(x) = \{y \in F(x) : \ \mathrm{dist}\,(y, C) = \nu_{F,C}(x)\} \quad \text{(farthest point mapping)}.$$

Under mild assumptions, it should be

$$\overline{\mathrm{co}}^* \bigcup_{y\in\overline{F}(\bar{x})} \widehat{\mathrm{D}}F^*(\bar{x}, y)\,(\partial\mathrm{dist}\,(\cdot, C)\,(y)) \subseteq \partial\nu_{F,C}(\bar{x}).$$

Problem 1: If F is C-concave, under proper assumptions on the stability of $\overline{F}$ near $\bar{x}$, is it true that

$$\overline{\mathrm{co}}^* \bigcup_{y\in\overline{F}(\bar{x})} \widehat{\mathrm{D}}F^*(\bar{x}, y)\,(\partial\mathrm{dist}\,(\cdot, C)\,(y)) = \partial\nu_{F,C}(\bar{x})\ ?$$

Problem 2: Extend such a characterization to the non convex/concave case by using the Fréchet subdifferential (or other subdifferentials).

Conclusions

- Set-valued inclusions are problems which emerge in the robust approach to uncertain optimization and in vector optimization.
- Some perspectives on their solution stability can be drawn by tools and techniques of variational analysis.
- The resulting theoretical picture seems to be different from that for generalized equations (e.g. metric increase replaces metric regularity, concavity replaces convexity).
- Many aspects of the topic still remain to be investigated and clarified.

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