VECTOR OPTIMIZATION WITH DOMINATION STRUCTURES: VARIATIONAL PRINCIPLES AND APPLICATIONS

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EVP, Ekeland & Turnbull (1983), Ekeland (1972)

Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function which is lower semicontinuous, bounded from below, and not identically equal to $+\infty$. For any $\varepsilon > 0$, for any ε-minimum solution x_0 of φ, for any $\lambda > 0$, there exists some point $x_* \in \operatorname{dom} \varphi$ such that

(i) $\varphi(x_*) \leq \varphi(x_0)$;

(ii) $d(x_0, x_*) \leq \lambda$;

(iii) $\varphi(x) + (\varepsilon/\lambda) d(x_*, x) > \varphi(x_*), \ \forall\, x \neq x_*$.

(i) and (ii) can be written as $\varphi(x_*) + (\varepsilon/\lambda) d(x_*, x) \leq \varphi(x_0)$.

Consider $S : X \rightrightarrows X$ with

$$S(x) := \Big\{ y \in X \mid \varphi(y) \leq \varphi(x) - (\varepsilon/\lambda) d(x, y) \Big\}.$$

Then, $x_* \in S(x_0)$ and $S(x_*) = \{x_*\}$.

EVP via Change Model

1. Status quo: the bundle of activities x in the previous period.
2. Stay or change: $x \curvearrowright x$ or $x \curvearrowright y$
3. Motivation to change: payoffs
$$M(x, y) = f(y) - f(x)$$
4. Resistance to change: costs
$$R(x, y) = C(x, y) - C(x, x)$$
5. Worthwhile to change
$$M(x, y) \geq r\, R(x, y)$$
6. Variational traps $W(x_*) = \{x_*\}$
$$\begin{aligned} & W(x_*) = \{x \ : \ \text{worthwhile to change } x_* \curvearrowright x\} = \{x_*\} \\ \iff & \forall x \neq x_*,\ M(x_*, x) < r\, R(x_*, x) \\ \iff & x_* \ \text{ is a maximal solution of } g(x) = f(x) - r\, C(x_*, x). \end{aligned}$$

Cost Requirements

1. Costs to be able to stay are zero, $C(x, x) = 0$ for all $x \in X$.
2. Costs to be able to change are positive, $C(x, y) > 0$ for all $x \neq y$.
3. The cost to be able to change from x to y is not necessarily equal to the cost to be able to change from y to x.
4. Direct costs are cheaper than or as equal as indirect costs, $C(x, y) \leq C(x, z) + C(z, y)$ for all $x, y, x \in X$.

Examples: $C_1(x, y) = |x - y|$ and $C_2(x, y) = y - x$ if $y \geq x$ and 1 otherwise.

Quasimetrics and Metric Spaces

A quasi-metric space is a set X equipped with a function $q : X \times X \longmapsto \mathbb{R}_+ := [0, \infty)$ on $X \times X$ having the following three properties:

(q1) $q(x, y) \geq 0$ (non-negativity);

(q2) if $x = y$, then $q(x, y) = 0$ (equality implies indistinct);

(q3) $q(x, z) \leq q(x, y) + q(y, z)$ (triangularity).

If, in addition, it satisfies the symmetry property $q(x, x') = q(x', x)$ for all $x, x' \in X$, then q is a metric. We denote by (X, q) the space X with the quasi-metric q. Quasi-metrics were introduced by Hausdorff in 1914 in his famous "*Grundzüge der Mengenlehre*" which is the foundation of the theory of topological and metric spaces.

Examples on Quasimetric Spaces

- The Sorgenfrey quasimetric on $\mathbb{R}$ is defined by $q(x, y) = y - x$ if $y \geq x$ and $q(x, y) = 1$ otherwise.
- The quasimetric on $\mathbb{R}$ is defined by $q(x, y) = \max(y - x, 0)$.
- The real half-line quasimetric is defined by $q(x, y) = \max(0, \ell n \frac{y}{x})$ on the set of strictly positive reals.
- A circular railroad line moves only in a counterclockwise direction around a circular track, represented by the unit circle $\mathcal{S}^1$. The circular-railroad quasi-metric from any point, $x \in \mathcal{S}^1$, to any other point, $y \in \mathcal{S}^1$, is simply the counterclockwise circular arc length from x to y in $\mathcal{S}^1$.
- Consider $X := \{u \in L^1(\Omega, \mathbb{R}^p) : \|u\|_\infty \leq 1\}$ equipped with the weak L^1- topology. The dissipation distance related to the energetic formulation of energetic models for rate- independent systems is defined by $q(u_1, u_2) = \|u_1 - u_2\|_{L^1}$.
- The Minkowski gauge function is defined on $\mathbb{R}^n$ by $q_B(x, y) = \inf\{\alpha > 0 : y - x \in \alpha B\}$, where B is convex and compact.

Definitions

Given a quasimetric space (X, q), i.e., a nonempty set X equipped with a quasi-metric q, we say that a sequence $\{x_n\}$

(i) is forward convergent to x_*, if $\lim_{n \to +\infty} q(x_*, x_n) = 0$;

(ii) is a forward Cauchy sequence, if for each $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $q(x_m, x_n) < \varepsilon$, for $m \geq n \geq \bar{n}$;

(iii) the space (X, q) is forward Hausdorff, if every forward convergent sequence has an unique forward limit;

(iv) the space (X, q) is forward-forward complete, if every forward Cauchy sequence is forward convergent.

Since a quasimetric is not symmetric, there are the corresponding backward concepts.

Remarks

Let (X, q) be a quasi-metric space and A be a nonempty subset of X. Then:

• a sequence $\{x_n\}$ is forward convergent in (X, q) is not necessary forward Cauchy;

• if $\{x_n\}$ is both forward and backward convergent to x_*, then x_* is the only limit point of $\{x_n\}$ of any kind;

• if $\{x_n\}$ has more than one forward limit points, then $\{x_n\}$ has no backward limit point;

• if $\{x_n\}$ is forward convergent to a and backward convergent to b, then $q(a, b) = 0$;

• if $\{x_n\}$ is forward convergent to a and $q(a, b) = 0$, then it is backward convergent to b;

• the function $q(\cdot, A) : X \to \mathbb{R}_+ = [0, \infty)$ defined by

$$q(x, A) := \inf_{u \in A} q(x, u)$$

is forward lower semicontinuous.

Example

Let X be a closed unit interval $[0,1]$ with the quasi-metric on X defined by

$$q(x,y) = \begin{cases} x - y & \text{if } x \geq y, \\ 1 & \text{if } x < y. \end{cases}$$

Consider the sequence $\{x_n\}$ where $x_n = 1/n$. Since $q(x_n, x_m) = 1/n - 1/m < 1/n$ for all $m, n \in \mathbb{N}$ with $m > n$, $\{x_n\}$ is a forward Cauchy sequence.

Take an arbitrary number $\bar{x} \in (0,1]$. For any integer $n \in \mathbb{N}$ with $n > 1/\bar{x}$, one has $x_n = 1/n < \bar{x}$ and thus $q(x_n, \bar{x}) = 1$, i.e., $\bar{x}$ is not a forward limit of $\{x_n\}$.

Obviously, 0 is the only forward limit of $\{x_n\}$.

Binary Relations Induced from Domination Structures

Let Y be a linear space partially equipped with a domination set $0 \in D \neq \emptyset$. The binary relation $\leq_C$ is defined by:

$$y_1 \leq_D y_2 \ \text{ if and only if } \ y_1 \in y_2 - D \ \text{ for all } \ y_1, y_2 \in Y.$$

When C is a proper, closed, convex, and pointed cone C, the binary relation $\leq_D$ is a partial order.

Assume that each element of Y has its own domination set. Then, the set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$ is called a domination structure in the linear space Y. We introduce the following binary relations:

(i) The nondomination binary relation $\leq_N$ is defined by

$$v \leq_N y :\Longleftrightarrow y \in v + \mathcal{D}(v).$$

(ii) The efficiency binary relation $\leq_E$ is defined by

$$v \leq_E y :\Longleftrightarrow v \in y - \mathcal{D}(y).$$

Nondominated and Efficient Solutions

Let $f : X \to Y$ be a mapping from a nonempty set to a linear space, and let $\mathcal{D} : Y \rightrightarrows Y$ be a domination structure in the image space Y. Given $\bar{x} \in \operatorname{dom} f$, we say that:

(i) $\bar{x}$ is a nondominated solution of f w.r.t. $\mathcal{D}$, or a $\mathcal{D}$-nondominated solution, or a $\leq_N$-minimal solution, if

$$\forall x \in \operatorname{dom} f,\ f(x) \leq_N f(\bar{x}) \Longrightarrow f(\bar{x}) \leq_N f(x).$$

(ii) $\bar{x}$ is a efficient solution of f w.r.t. $\mathcal{D}$, or a $\mathcal{D}$-efficient solution, or a $\leq_E$-minimal solution, if

$$\forall x \in \operatorname{dom} f,\ f(x) \leq_E f(\bar{x}) \Longrightarrow f(\bar{x}) \leq_E f(x).$$

Should I change? or should I regret to have changed?.

Efficiency binary relation: should I change? Yes, if the advantages to move from y to v (change rather than stay) in the payoff space is $\mathbb{A}(y, v) := y - v = f(x) - f(u) = A(x, u) \in \mathcal{D}(y) = \mathcal{D}(f(x))$, which means that there are ex ante advantages to move from y to v, i.e., from x to u.

Nondomination binary relation: should I regret to have changed? No, if the agent would prefer to change from y to v after moving, i.e., to go from x to u rather than to stay at y provided that the new amount of pains $v = f(u)$ is perceived ex post as lower than the old amount of pains $y = f(x)$.

Dancs-Hegedüs-Medvegyev Fixed Point Theorem (1983)

Let (X, d) be a complete metric space, and let $\Phi : X \rightrightarrows X$ be a set-valued mapping satisfying the following conditions:

(A1) $x \in \Phi(x)$ for all $x \in X$;

(A2) $x_2 \in \Phi(x_1) \Longrightarrow \Phi(x_2) \subseteq \Phi(x_1)$ for all $x_1, x_2 \in X$;

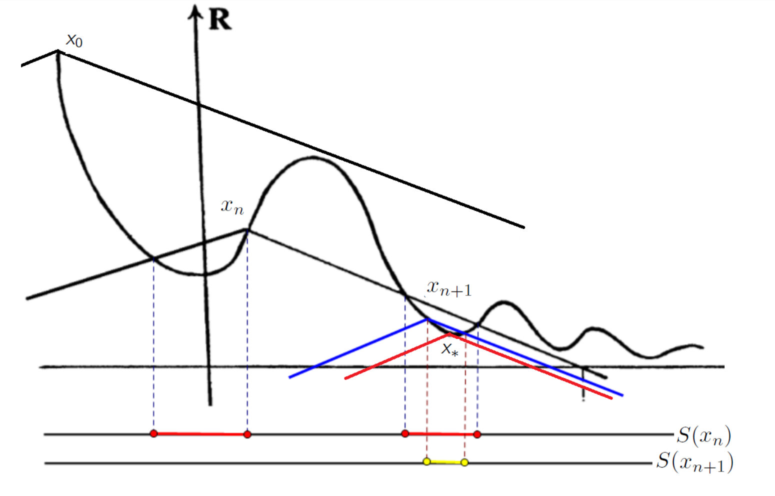
(A3) For each generalized Picard sequence $\{x_n\}$ of Φ, $d(x_n, x_{n+1}) \to 0$;

(A4) $\Phi(x)$ is a closed set for all $x \in X$;

Then, for every starting point $x_0 \in X$, there is a convergent sequence $\{x_n\} \subseteq X$ whose limit x_* is a fixed point of Φ, i.e., $\Phi(x_*) = \{x_*\}$.

Theorem (DHMFPT) $\Longrightarrow$ Theorem (EVP)
where $S_{\tau,d}(x) = \Big\{ y \in X |\ \varphi(y) \leq \varphi(x) - \tau d(x, y) \Big\}$.

Proof. Set $S(x_{n+1}) := \left\{x \in X \mid \varphi(x) \leq \varphi(x_n) - d(x_n, x)\right\}$.



We have $S(x_0) \supseteq S(x_1) \supseteq \ldots \supseteq S(x_n) \supseteq S(x_{n+1}) \supseteq \ldots \supseteq S(x_*) = \{x_*\}$.

A Revised Version of Fixed Point Theorem, BG (2021)

Let (X, d) be a complete metric space, $x_0 \in X$ and $\Phi : X \rightrightarrows X$ be a dynamic system. Suppose that each generalized Picard sequence (x_n) of Φ whose starting point is x_0 satisfies the following conditions:

(B1) $\Phi(x_{n+1}) \subseteq \mathrm{cl}\, \Phi(x_n)$ for all $n \in \mathbb{N}$.

(B2) $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$.

(B3) If $x_n \to x$ and $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, then $\Phi(x) \subseteq \mathrm{cl}\, \Phi(x_n)$ for all $n \in \mathbb{N}$.

Then, there is a fixed point $x_* \in \mathrm{cl}\, \Phi(x_0) \cup \{x_0\}$ of the system Φ; i.e., $\Phi(x_*) \subseteq \{x_*\}$. Assume furthermore that $\Phi(x_*) \neq \emptyset$, then $\Phi(x_*) = \{x_*\}$.

Examples

Let (X_1, d) be the complete metric space given by the set $X_1 = \{1/n : n \in \mathbb{N}\backslash\{0\}\} \cup \{0\}$ and the metric $d(x_1, x_2) = |x_1 - x_2|$, for all $x_1, x_2 \in X_1$. Consider the point $x_0 = 1$ and the following dynamic system $\Phi_1 : X_1 \rightrightarrows X_1$:

$$\Phi_1(x) = \begin{cases} \{0\} & \text{if} \quad x = 0, \\ \{1/m : m \in \mathbb{N}\backslash\{0\}, m > n\} & \text{if} \quad x = 1/n \text{ and } n \text{ is even}, \\ \{1/m : m \in \mathbb{N}\backslash\{0\}, m > n\} \cup \{0\} & \text{if} \quad x = 1/n \text{ and } n \text{ is odd}. \end{cases}$$

As all assumptions are satisfied, it follows that there exists a point $\bar{x} \in X_1$ such that $\Phi_1(\bar{x}) \subseteq \{\bar{x}\}$. Obviously, $\bar{x} = 0$ and $\Phi_1(0) = \{0\}$. Notice that $1/n \notin \Phi_1(1/n)$, for all $n \in \mathbb{N}\backslash\{0\}$, $\Phi_1(1/n)$ is not closed provided that n is even, and $1/m \in \Phi_1(1/n)$, $\Phi_1(1/m) \not\subseteq \Phi_1(1/n)$ as long as n is even and $m > n$ is odd. Thus, assumptions in the original fixed point theorem are not fulfilled and so it cannot be applied. For instance, for all $n \in \mathbb{N}\backslash\{0\}$, $\frac{1}{2n+1} \in \Phi_1(\frac{1}{2n})$ but $\Phi_1(\frac{1}{2n+1}) \not\subseteq \Phi_1(\frac{1}{2n})$.

Examples

Assumption (B2) cannot be dropped. Indeed, the dynamic system $\Phi_2 : X_1 \rightrightarrows X_1$, $\Phi_2(x) = X_1$ for all $x \in X_1$, satisfies assumptions (B1) and (B3), but it has not any fixed point. It is obvious that hypothesis (B2) is not fulfilled.

Hypothesis (B3) cannot be removed. For instance, the dynamic system $\Phi_3 : X_1 \rightrightarrows X_1$, $\Phi_3(0) = X_1$ and $\Phi_3(1/n) = \{1/m : m \in N \backslash \{0\}, m > n\}$ fulfills (B1) and (B2), but is doesn't satisfy (B3). Clearly, Φ_3 has not any fixed point.

Assumption (B1) is also needed. For instance, let

$$X_2 = \left\{ s_n := \sum_{m=1}^{n} 1/m : n \in N \backslash \{0\} \right\}.$$

It is obvious that (X_2, d) is a complete metric space. The dynamic system $\Phi_3 : X_2 \rightrightarrows X_2$, $\Phi_3(s_n) = \{s_{n+1}\}$, for all $n \in N \backslash \{0\}$ fulfills hypotheses (B2) and (B3), but it has not any fixed point. It is easy to check that $\Phi_3(s_{n+1}) \not\subseteq \Phi_3(s_n)$, for all $n \in N \backslash \{0\}$.

Nonlinear Scalarization Function, GKNR (2017)

Let Y be a real linear space, D be a nonempty set in Y and k be a nonempty element in Y. D is said to be a domination set if $0 \in D$.

The vectorial closure of D in the direction k is defined by

$$\mathrm{vcl}_k D := \{y \in Y \mid \forall \lambda > 0, \exists t \in [0, \lambda], y + tk \in D\}.$$

The function $\varphi_D^k : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_D^k(y) := \inf\{t \in \mathbb{R} : \ y \in tk - D\} \text{ with } \inf \emptyset = +\infty.$$

is called Gerstewitz' nonlinear scalarization function.

Assume that D is free-disposal in the direction k; i.e., $D + \mathrm{cone}(k) \subseteq D$ and $D \cap -\mathrm{cone}(k) = \{0\}$. Then, the following hold:

- $\mathrm{dom}\, \varphi_D^k = D + \mathbb{R}\, k$.
- $\forall y \in Y, \forall t \in \mathbb{R}, \varphi_D^k(y + tk) = \varphi_D^k(y) + t$.
- $\varphi_D^k(y) \leq t \iff y \in t - \mathrm{vcl}_k D$.

Some Properties of Gerstewitz' Scalarizing Functions

Let $\emptyset \neq D \subseteq Y$ and $k \in Y \setminus \{0\}$ satisfy $D + [0, +\infty)k \subseteq D$. Then the following hold:

(a) φ_D^k is l.s.c. over its domain $\operatorname{dom} \varphi_D^k = \mathbb{R}k - D$ iff D is a closed set. Moreover, its t-level set is given by

$$\{y \in Y \mid \varphi_D^k(y) \leq t\} = tk - D, \ \forall\, t \in \mathbb{R}$$

and $\varphi_D^k(y + tk) = \varphi_D^k(y) + t, \ \forall\, y \in Y, \ \forall\, t \in \mathbb{R}$.

(b) φ_D^k is convex if and only if the set D is convex, and φ_D^k is positively homogeneous if and only if D is a cone.

(c) φ_D^k is proper if and only if D does not contain lines parallel to k.

(d) φ_D^k is finite-valued, i.e. $\operatorname{dom} \varphi_D^k = Y$, if and only if $\mathbb{R}k - D = Y$ and D does not contain lines parallel to k.

(e) Given $B \subseteq Y$. φ_D^k is B-monotone if and only if $D + B \subseteq D$.

(f) φ_D^k is subadditive if and only if $D + D \subseteq D$.

Let (X, q) be a quasimetric space, let Y be a linear space equipped with a variable domination structure $\mathcal{D} : Y \rightrightarrows Y$, and let $f : X \to Y$ be a vector-valued mapping. Given $k \in Y \setminus \{0\}$, $x_0 \in X$, $y_0 := f(x_0)$, $\Theta := \mathcal{D}(y_0)$, and $\varepsilon \geq 0$, we consider the set-valued mapping $W : X \rightrightarrows X$ defined by

$$W(x) := \left\{u \in X \;\middle|\; f(x) - f(u) - \sqrt{\varepsilon} q(x, u) k \in \mathcal{D}(f(x_0))\right\}$$

and the extended-real-valued function $\psi : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\psi(x) := \varphi_{\Theta,k}(f(x) - f(x_0)).$$

Efficiency EVP, BMST (2020)

Impose the following assumptions:

(E1) (boundedness condition) The scalarized function ψ is bounded from below over $W(x_0)$.

(E2) (limiting monotonicity condition) For every infinite nonconstant generalized Picard sequence $\{x_n\}$ of the set-valued mapping W the convergence of the series $\sum_{n=0}^{\infty} q(x_n, x_{n+1})$ yields the existence of x_* such that

$$\forall n \in \mathbb{N},\ x_* \in W(x_n).$$

(E3) (scalarization condition) Θ is k-vectorial closed with $0 \in \Theta$, $\Theta + \Theta \subseteq \Theta$, $\Theta + \operatorname{cone}(k) \subseteq \Theta$, and $\Theta \cap (-\operatorname{cone}(k)) = \{0\}$.

Then, there exists $x_* \in W(x_0)$ satisfying the inclusion

$$W(x_*) \subseteq \overline{\{x_*\}} := \{u \in X \mid q(x_*, u) = 0\}.$$

Efficiency EVP, BMST (2021)

If in addition the condition

(E4) (uniqueness limit condition) (X, q) is forward-Hausdorff

is satisfied, then the conclusions of this theorem reduce to

(i) $f(x_0) - f(x_*) - \sqrt{\varepsilon} q(x_0, x_*) k \in \mathcal{D}(f(x_0))$ and
(ii) $\forall x \in X \setminus \{x_*\}, f(x_*) - f(x) - \sqrt{\varepsilon} q(x_*, x) k \notin \mathcal{D}(f(x_0))$.

Furthermore, imposing the domination condition

(E5) $\mathcal{D}(f(x_*)) \subseteq \mathcal{D}(f(x_0))$

ensures that

$$\forall x \in X \setminus \{x_*\}, f(x_*) - f(x) - \sqrt{\varepsilon} q(x_*, x) k \notin \mathcal{D}(f(x_*)).$$

If finally the starting point x_0 is an εk-efficient solution of f w.r.t. $\mathcal{D}$, then x_* can be chosen so that in addition to (i) and (ii) we have

(iii) $q(x_0, x_*) \le \sqrt{\varepsilon}$.

A Sketch of the Proof

Starting with x_0, we assume that x_n is given. If $W(x_n) = \{x_n\}$, then $x_* = x_n$ satisfies the desired conclusion. Otherwise, choose $x_{n+1} \in W(x_n) \setminus \{x_n\}$ satisfying

$$\psi(x_{n+1}) \leq \inf_{u \in W(x_n)} \psi(u) + \frac{1}{2^{n+1}}.$$

It is obvious that such an element x_{n+1} exists due to the boundedness from below of the function ψ assumed in (E1). We aim at verifying that the consecutively different generalized Picard sequence $\{x_n\}$ forward-converges to the desired element by splitting the proof into several steps.

A Sketch of the Proof

- If $u \in W(x)$, then $f(u) \leq_\Theta f(x)$ and $W(u) \subseteq W(x)$.
- For every $u \in W(x_n)$ we have the estimate

$$\forall n \in \mathbb{N}, \forall x \in W(x_n),\ \sqrt{\varepsilon} q(x_n, u) \leq \frac{1}{2^n}.$$

- The series $\sum_{n=1}^{\infty} q(x_n, x_{n+1})$ is convergent.
- The inclusion in $W(x_*) \subseteq \overline{\{x_*\}} := \left\{u \in X \,\middle|\, q(x_*, u) = 0\right\}$ is satisfied.
- Imposing $(\mathrm{E4})$ gives us assertions (i) and (ii) of the theorem.
- The domination inclusion in $(\mathrm{E5})$ yields

$$\forall x \in X \setminus \{x_*\}, f(x_*) - f(x) - \sqrt{\varepsilon} q(x_*, x) k \notin \mathcal{D}(f(x_*)).$$

- If x_0 is an εk-efficient solution of f w.r.t. $\mathcal{D}$, we have (iii).

Illustration for the Efficiency EVP

Let $X := \mathbb{R}$ and $Y := \mathbb{R}^2$, and let $f : X \to Y$ be defined by

$$f(x) := \begin{cases} (x, 2^x - 1) & \text{if } \ x < 0, \\ (x, 1) & \text{if } \ x \geq 0. \end{cases}$$

The domination structure $\mathcal{D} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is given by

$$\mathcal{D}(y) := \begin{cases} \operatorname{cone} \left\{(1,0), (|y_1|, |y_2|)\right\} & \text{if } \ y_1 < 0 \ \text{ and } \ y_2 < 0, \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

Take $\varepsilon = 1$, $x_0 = 0$, $k = (1,1)$, and $d(x,u) = 1/2|x - u|$. In this case we have $\psi(x) = \varphi_{\mathbb{R}^2_+, k}$. It is easy to check that:

a) $W(0) = [-0.5, 0]$ and $W(-0.5) = \{-0.5\}$.
b) f is $\varphi_{\mathbb{R}^2_+, k}$-bounded from below.
c) f is not $\mathbb{R}^2_+$-lower semicontinuous since

$$\operatorname{lev}(f, 0_{\mathbb{R}^2}) = \left\{x \in X \;\middle|\; f(x) \in 0 - \mathbb{R}^2_+\right\} = (-\infty, 0)$$

is not a closed set in $\mathbb{R}$

Illustration for the Efficiency EVP

d) $f(-0.5) = (-0.5, -1 + 1/\sqrt{2})$, $f(0) = (0, 1)$, $\mathcal{D}(f(-0.5)) = \text{cone}\{(1, 0), (0.5, 1 - 1/\sqrt{2})\}$, and $\mathcal{D}(f(0)) = \mathbb{R}^2_+$. It is obvious that $f(-0.5) \leq_{\mathcal{D}(f(x_0))} f(0)$ and $\mathcal{D}(f(-0.5)) \subseteq \mathcal{D}(f(0))$, and hence condition (E5) is satisfied.

e) Condition (E2) holds since for any nonconstant generalized Picard sequences in W_0 (without loss of generality it can assumed that $x_n < 0$) we have that $W(x_n)$ is closed whenever $n \in \mathbb{N}$. Then, the existence of x_* follows from the classical Cantor theorem.

Vectorial EVP

Let $X \neq \emptyset$, $f : X \to Y$, $W : X \rightrightarrows X$ with
$W(x) := \{u \in E \mid f(u) + q(x,u)k \leq_D f(x)\}$, $x_0 \in \operatorname{dom} f$. Assume that

(D1) q is a quasi-metric on $W(x_0)$ and $\leq_D$ is a preorder.

(D2) φ_D^q is bounded from below on $f(W(x_0)) - f(x_0)$.

(D3) f is a strictly φ_D^k-decreasing lower-semi-continuous on $W(x_0)$ in the sense that for every Picard sequence (x_n) of W, one has

$$\begin{aligned} & \forall n \in \mathbb{N}, \varphi_D^k(f(x_{n+1})) < \varphi_D^k(f(x_n)) \\ \Longrightarrow \quad & \exists x_* \in X, q(x_n, x_*) \to 0 \ \text{ and } \ \forall n \in \mathbb{N}, f(x_*) \leq_D f(x_n). \end{aligned}$$

(D4) For each distinct Picard sequence $\{x_n\}$ of W, $q(x_n, x_*) \to 0$ and $q(x_n, y_*) \to 0$ imply $x_* = y_*$.

Then, these exists $x_* \in W(x_0)$ such that $W(x_*) = \{x_*\}$.

Remarks

In Theorem 5.1 in [Soleimani, JOTA 2014] and Theorem 3.8 [Bao et al, JCA 2017], the boundedness condition (E1): f is bounded from below; the condition (E2): f is $(k, \mathcal{D}(y_0))$-lower semicontinuity in the sense that $M(t) = \{x \in X : f(x) \in tk - \operatorname{cl} \mathcal{D}(y_0)\}$ is closed for all $t \in \mathbb{R}$; the scalarization condition (E3) is assumed for all domination sets $\mathcal{D}(y)$ for $y \in Y$.

In Theorem 3.12 [BEST, JCA 2017], the boundedness condition (E1): f is quasibounded from below w.r.t. $\mathcal{D}(y_0)$; the condition (E2): f is $\mathcal{D}(y_0)$-lower semicontinuity in the sense that $\operatorname{Lev}(y; f) = \{x \in X : f(x) \in y - \mathcal{D}(y_0)\}$ is closed for all $y \in y$; the scalarization condition (E3): $\mathcal{D}(y_0)$ is a proper, closed, convex and pointed cone.

The boundedness condition (E1) of the scalarized function ψ is equivalent to the existence of a real number m such that colorred

$$\psi(x) = \varphi(f(x)) > m \quad \text{for all} \quad x \in \operatorname{dom} f \quad \Longleftrightarrow \quad f(x) \notin mk - \mathcal{D}(y_\varepsilon).$$

On Boundedness Condition

The boundedness from below condition of φ_D^q on X is equivalent to

$$\exists \tau \in \mathbb{R}, \forall x \in X, f(x) \notin \tau k - D.$$

Consider $f : \mathbb{R} \to \mathbb{R}^2$ with

$$f(x) := \begin{cases} (0, -x) & \text{if } x \geq 0; \\ (x, 0) & \text{if } x < 0. \end{cases}$$

Let $D = \mathbb{R}^2_+$ and $k = (1, 1)$. We have $\varphi_D^k(f(x)) \equiv 0$ and thus bounded from below.

f is not quasibounded from below in the sense that there is a bounded set M such that

$$\forall x \in X, f(x) \in M + K.$$

Nondomination EVP, BMST (2020)

Let (X, q) be a quasimetric space, let Y be a linear space, let $\mathcal{D} : Y \rightrightarrows Y$ be a domination structure on Y with the nondomination relation $\leq_N$, let $k \in Y \setminus \{0\}$, and let $\Theta := \mathcal{D}(f(x_0))$. Given $x_0 \in X$ and $\varepsilon \geq 0$, define the set-valued mapping $W = W_{f,q,\varepsilon} : X \rightrightarrows X$ by

$$W(x) := \left\{ u \in X \;\middle|\; f(x) - \sqrt{\varepsilon} q(x, u) k - f(u) \in \mathcal{D}(f(u)) \right\}.$$

Consider also the extended-real-valued function $\psi : X \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\psi(x) := \varphi_{\Theta,k}(f(x) - f(x_0)).$$

Nondomination EVP, BMST (2021)

Impose the following assumptions:

(F1) (boundedness condition) ψ is bounded from below on $W(x_0)$; i.e., there exists $\tau \in \mathbb{R}$ such that $\psi(x) \geq \tau$ for all $x \in W(x_0)$.

(F2) (limiting monotonicity condition) for every generalized Picard sequence $\{x_n\}$ of the set-valued mapping W, the convergence of the series $\sum_{n=0}^{\infty} q(x_n, x_{n+1})$ yields the existence of x_* satisfying $x_* \in W(x_n)$ for all $n \in \mathbb{N}$.

(F3) (scalarization conditions)

- **(F3-a)** Θ is k-vectorially closed, $\Theta + \Theta \subseteq \Theta$, $\Theta + \mathbf{cone}\,(k) \subseteq \Theta$, and $\Theta \cap -\mathbf{cone}\,(k) = \{0\}$.
- **(F3-b)** $\forall x \in W(x_0)$, $\mathcal{D}(f(x)) + \mathbf{cone}\,(k) \subseteq \mathcal{D}(f(x))$.
- **(F3-c)** $\forall f(u) - f(w) \in \mathcal{D}(f(w))$, $\mathcal{D}(f(w)) + \mathcal{D}(f(u)) \subseteq \mathcal{D}(f(w))$.
- **(F3-d)** $\Theta^u \subseteq \Theta$, where $\Theta^u := \cup\{\mathcal{D}(f(x)) \ : \ x \in W(x_0)\}$.

Nondomination EVP, BMST (2021)

Then, there exists $x_* \in W(x_0)$ such that $W(x_*) \subseteq \overline{\{x_*\}}$, where

$$\overline{\{x_*\}} := \{u \in X \mid q(x_*, u) = 0\}.$$

Assuming furthermore that

(F4) (uniqueness condition) the forward-limit of a forward convergent sequence in the quasimetric space (X, q) is unique.

Then, the conclusions of the theorem can be written as

(i) $f(x_0) - \sqrt{\varepsilon} q(x_0, x_*) - f(x_*) \in \mathcal{D}(f(x_*))$,

(ii) $\forall x \neq x_*,\ f(x_*) - \sqrt{\varepsilon} q(x_*, x)k - f(x) \notin \mathcal{D}(f(x))$.

If the starting point x_0 is an εk–nondominated solution of f w.r.t. $\mathcal{D}$, then we have in addition to (i) and (ii) that x_* satisfies

(iii) $q(x_0, x_*) \leq \sqrt{\varepsilon}$.

Remarks

Theorem 3.12 in [BEST, JCA 2017] provides a nondominated version of EVP. Under appropriate assumptions, there exists an element $\bar{x} \in \operatorname{dom} f$ such that the following conditions hold:

(i) $s(f(\bar{x})) + \sqrt{\varepsilon} d(\bar{x}, x_\varepsilon) \leq s(f(x_\varepsilon))$.

(ii) $d(\bar{x}, x_\varepsilon) \leq \sqrt{\varepsilon}$.

(iii) $\bar{x}$ is an exact solution of the scalarized function defined by $f_{\bar{x}} := s \circ f + \sqrt{\varepsilon} d(\bar{x}, \cdot)$,

where s is an extended version of the Gerstewitz scalarization function $s : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by the formula

$$s(y) := \inf \left\{ t \in \mathbb{R} \;\middle|\; y \in a + tk - \mathcal{D}(y) \right\}.$$

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Thank you!