GENERALIZED NEWTON METHODS
VIA VARIATIONAL ANALYSIS

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talk given at the
INTERNATIONAL WORKSHOP ON OPTIMIZATION
AND OPERATOR THEORY

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Haifa, Israel

November 2021
NEWTONIAN METHODS FOR SMOOTH FUNCTIONS

First consider the unconstrained optimization problem

\[
\text{minimize } \varphi(x) \text{ subject to } x \in \mathbb{R}^n
\]

with \( C^2 \)-smooth objective function \( \varphi \). The classical Newton method exhibits the local convergence with a quadratic rate provided that \( \nabla^2 \varphi(x) \) is positive-definite. To achieve the global convergence, various line search procedures are used

\[
x^{k+1} := x^k + \tau_k d^k \quad \text{with} \quad -\nabla \varphi(x^k) = H_k d^k
\]

where \( H_k \) is an appropriate approximation of the Hessian \( \nabla^2 \varphi(x) \) for quasi-Newton methods. The Levenberg-Marquardt method

\[
H_k := \nabla^2 \varphi(x^k) + \mu_k I \quad \text{with} \quad \mu_k := c\|\nabla \varphi(x^k)\|
\]

works when \( \nabla^2 \varphi(x^k) \) is merely positive-semidefinite.
MAJOR GOALS

Replacing the Hessian $\nabla^2 \varphi$ by its coderivative-based generalized Hessian (second-order subdifferential) $\partial^2 \varphi$, pursue the following:

- Design and justify the **globally convergent generalized damped Newton method** with the backtracking line search for unconstrained problems of $C^{1,1}$ optimization.
- Design and justify the **globally convergent Levenberg-Marquardt method** with the backtracking line search for unconstrained problems of $C^{1,1}$ optimization.
- Using forward-backward envelopes, extend both coderivative-based generalized Newton methods to problems of **convex composite optimization** encompassing problems with constraints.
- Solving *Lasso problems* by the developed generalized Newton algorithms with **numerical experiments** and comparison with other **first-order and second-order** algorithms of optimization.
See [M06, M18, RW98] for more details and references. The (limiting) normal cone to $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$ from

$$N_{\Omega} (\bar{x}) := \left\{ v \mid \exists x_k \to \bar{x}, \alpha_k \geq 0, w_k \in \Pi_{\Omega} (x_k), \alpha (x_k - w_k) \to v \right\}$$

where $\Pi_{\Omega}$ stands for the Euclidean projection. The coderivative of $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{y}) \in \text{gph} F$

$$D^* F(\bar{x}, \bar{y})(v) := \left\{ u \in \mathbb{R}^n \mid (u, -v) \in N_{gph F}(\bar{x}, \bar{y}) \right\}, \quad v \in \mathbb{R}^m.$$ 

When $F: \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$-smooth, then

$$D^* F(\bar{x})(v) = \left\{ \nabla F(\bar{x})^* v \right\}, \quad v \in \mathbb{R}^m,$$

via the adjoint/transpose Jacobian matrix. The (first-order) subdifferential of $\varphi: \mathbb{R}^n := (-\infty, \infty]$ at $\bar{x} \in \text{dom} \varphi$ [M76]

$$\partial \varphi(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid (v, -1) \in N_{epi \varphi}(\bar{x}, \varphi(\bar{x})) \right\}.$$ 

Despite their nonconvexity these constructions enjoy full calculus based on the variational/extremal principles of variational analysis.
The second-order subdifferential, or generalized Hessian of $\varphi: \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \text{dom } \varphi$ for $\bar{v} \in \partial \varphi(\bar{x})$ is defined as [M92]

$$\partial^2 \varphi(\bar{x}, \bar{v})(u) := \left(D^* \partial \varphi\right)(\bar{x}, \bar{y})(u), \quad u \in \mathbb{R}^n.$$  

If $\varphi$ is $C^2$-smooth around $\bar{x}$, then

$$\partial^2 \varphi(\bar{x})(u) = \left\{\nabla^2 \varphi(\bar{x})u\right\}, \quad u \in \mathbb{R}^n.$$  

If $\varphi$ of class $C^{1,1}(C^1$ with Lipschitz gradient) around $\bar{x}$, then

$$\partial^2 \varphi(\bar{x})(u) = \partial \langle u, \nabla \varphi(\bar{x}) \rangle, \quad u \in \mathbb{R}^n.$$  

It is realized that the generalized Hessian $\partial^2 \varphi$ enjoys well-developed second-order calculus and can be viewed as an appropriate replacement of the Hessian $\nabla^2 \varphi$ for nonsmooth problems. $\partial^2 \varphi$ is fully computed in terms of the given data for broad classes of problems in optimization and variational analysis.
Algorithm 1 Coderivative-based damped Newton algorithm for $\mathcal{C}^{1,1}$

Input: $x^0 \in \mathbb{R}^n$, $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0, 1)$

1: for $k = 0, 1, \ldots$ do
2: If $\nabla \varphi(x^k) = 0$, stop; otherwise go to the next step
3: Choose $d^k \in \mathbb{R}^n$ such that $-\nabla \varphi(x^k) \in \partial^2 \varphi(x^k)(d^k)$
4: Set $\tau_k = 1$
5: while $\varphi(x^k + \tau_k d^k) > \varphi(x^k) + \sigma \tau_k \langle d^k, \nabla \varphi(x^k) \rangle$ do
6: set $\tau_k := \beta \tau_k$
7: end while
8: Set $x^{k+1} := x^k + \tau_k d^k$
9: end for

The main assumption for the well-posedness and global convergence

$$(PD) \quad \text{generalized Hessian } \partial^2 \varphi \text{ is positive-definite on } \mathbb{R}^n.$$
DEFINITION (Pol-Roc98) For $\varphi : \mathbb{R}^n \to \mathbb{R}$, a point $\bar{x} \in \text{dom } \varphi$ is a tilt-stable local minimizer with modulus $\ell$ if there is $\gamma$ such that

$$M_\gamma : v \mapsto \arg\min\{\varphi(x) - \langle v, x \rangle \mid x \in IB_\gamma(\bar{x})\}$$

is single-valued and Lipschitz continuous around the origin in $\mathbb{R}^n$ with constant $\ell$ and such that $M_\gamma(0) = \{\bar{x}\}$.

Theorem (Pol-Roc98) Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ is prox-regular and subdifferentially continuous [RW98] at $\bar{x}$ for $\bar{v} \in \partial \varphi(\bar{x})$ (this holds, in particular, for $C^{1,1}$ and for convex functions). Then $\bar{x}$ is tilt stable local minimizer of $\varphi$ for $\bar{v}$ if and only if

$$\partial^2 \varphi(\bar{x}, \bar{v})$$

is positive-definite.

By now we have complete characterizations of tilt stability with precise formulas for computing the best modulus bounds for major classes problems in constrained optimization and optimal control.
WELL-POSEDNESS AND CONVERGENCE OF ALGORITHM 1

Theorem[KMPT21] Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be of class $C^{1,1}$ under the fulfillment of (PD). Then whenever $\partial \varphi(x) \neq 0$ there is $d \neq 0$ with

$$-\nabla \varphi(x) \in \partial^2 \varphi(x)(d) \quad \text{and} \quad \langle \varphi(x), d \rangle < 0.$$ 

Thus for each $\sigma \in (0, 1)$ there exists $\delta > 0$ such that

$$\varphi(x + \tau d) \leq \varphi(x) + \sigma \tau \langle \nabla \varphi(x), d \rangle \quad \text{whenever} \quad \tau \in (0, \delta).$$

Furthermore, for any starting point $x^0$, each limiting point $\bar{x}$ of the sequence of iterates $\{x^k\}$ is a tilt-stable local minimizer of $\varphi$ satisfying the following conditions:

- The convergence rate of the sequence $\{\varphi(x^k)\}$ is at least $Q$-linear.
- The convergence rates of both sequences $\{x^k\}$ and $\{\|\nabla \varphi(x^k)\|\}$ are at least $R$-linear.
SUPERLINEAR GLOBAL CONVERGENCE OF ALGORITHM 1

Definition [Gfrerer-Outrata21] A mapping $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is semismooth$^*$ at $(\bar{x}, \bar{y}) \in \text{gph} F$ if

$$\langle u^*, u \rangle = \langle v^*, v \rangle \quad \text{for all} \quad (v^*, u^*) \in \text{gph} \, D^* F((\bar{x}, \bar{y}); (u, v)).$$

For single-valued and locally Lipschitzian mappings, this reduces to the semismooth property if $F$ is directionally differentiable.

Theorem [KMPT21] In the setting of the previous theorem, suppose that $\nabla \varphi(\bar{x})$ is semismooth$^*$ at $\bar{x}$. Then $\{x^k\}$ $Q$-superlinearly converges to $\bar{x}$ provided that either $\nabla \varphi$ is directionally differentiable at $\bar{x}$, or $\sigma \in (0, 1/(2\ell \kappa))$, where $\kappa$ is a modulus of tilt stability of $\bar{x}$ and $\ell$ is a Lipschitz constant of $\nabla \varphi$ around $\bar{x}$. Moreover, in this case the sequence $\{\varphi(x^k)\}$ converges $Q$-superlinearly to $\varphi(\bar{x})$, and the sequence $\{\nabla \varphi(x^k)\}$ converges $Q$-superlinearly to 0 as $k \to \infty$.
LEVENBERG-MARQUARDT METHOD IN $C^{1,1}$ OPTIMIZATION

The (PD) assumption is now replaced by

\[(PSD)\] generalized Hessian $\partial^2 \varphi$ is positive-semidefinite on $\mathbb{R}^n$.

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Algorithm 2 Levenberg-Marquardt algorithm for $C^{1,1}$ functions

**Input:** $x^0 \in \mathbb{R}^n$, $c > 0$, $\sigma \in (0, \frac{1}{2})$, $\beta \in (0, 1)$

1: for $k = 0, 1, \ldots$ do
2: If $\nabla \varphi(x^k) = 0$, stop; else let $\mu_k := c\|\nabla \varphi(x^k)\|$ and go to Step 3
3: Choose $d^k \in \mathbb{R}^n$ such that $-\nabla \varphi(x^k) \in \partial^2 \varphi(x^k)(d^k) + \mu_k d^k$
4: Set $\tau_k = 1$
5: while $\varphi(x^k + \tau_k d^k) > \varphi(x^k) + \sigma \tau_k \langle \nabla \varphi(x^k), d^k \rangle$ do
6: set $\tau_k := \beta \tau_k$
7: end while
8: Set $x^{k+1} := x^k + \tau_k d^k$
9: end for
WELL-POSEDNESS AND CONVERGENCE OF ALGORITHM 2

Theorem[KMPT21] Let $\varphi$ be of class $C^{1,1}$ under the fulfillment of (PSD). If $\partial \varphi(x) \neq 0$ and $\varepsilon > 0$, then there is $d \neq 0$ with

$$-\nabla \varphi(x) \in \partial^2 \varphi(x)(d) + \varepsilon d \quad \text{and} \quad \langle \varphi(x), d \rangle < 0.$$  

Thus for each $\sigma \in (0, 1)$ there exists $\delta > 0$ such that

$$\varphi(x + \tau d) \leq \varphi(x) + \sigma \tau \langle \nabla \varphi(x), d \rangle \quad \text{whenever} \quad \tau \in (0, \delta).$$

Furthermore, any starting point $x^0$ produces iterates $\{x^k\}$ such that the sequence of values $\{\varphi(x^k)\}$ is monotonically decreasing and all the limiting points of $\{x^k\}$ satisfy the stationary condition.
DEFINITION [M93, RW98] A mapping $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is metrically regular around $(\bar{x}, \bar{y}) \in \text{gph } F$ if there exist $\mu > 0$ and neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$\text{dist}(x; F^{-1}(y)) \leq \mu \text{dist}(y; F(x))$$

for all $(x, y) \in U \times V$, where $F^{-1}(y) := \{x \in \mathbb{R}^n \mid y \in F(x)\}$.

Coderivative/Mordukhovich criterion: If a set-valued mapping $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is of closed-graph around $(\bar{x}, \bar{y})$, then its metric regularity around this point is equivalent to

$$D^* F(\bar{x}, \bar{y})(0) = \{0\}.$$
RATES OF CONVERGENCE FOR ALGORITHM 2

**THEOREM [KMPT21]** Let \( \bar{x} \) be a limiting point of the sequence of iterates in Algorithm 2. In addition to (PSD), suppose that \( \nabla \varphi \) is metrically regular around this point. Then \( \bar{x} \) is a tilt-stable local minimizer of \( \varphi \), and Algorithm 2 converges to \( \bar{x} \) with the convergence rates as follows:

- The sequence \( \{ \varphi(x^k) \} \) converges to \( \varphi(\bar{x}) \) at least \( Q \)-linearly.
- The sequences \( \{ x^k \} \) and \( \{ \nabla \varphi(x^k) \} \) converge at least \( R \)-linearly to \( \bar{x} \) and 0, respectively.
- The convergence rates of \( \{ x^k \} \), \( \{ \varphi(x^k) \} \), \( \{ \nabla \varphi(x^k) \} \) are at least \( Q \)-superlinear if \( \nabla \varphi \) is semismooth* at \( \bar{x} \) and either one of the following two conditions holds:
  (a) \( \nabla \varphi \) is directionally differentiable at \( \bar{x} \),
  (b) \( \sigma \in (0, 1/(2\ell\kappa)) \), where \( \kappa > 0 \) and \( \ell > 0 \) are moduli of metric regularity and Lipschitz continuity of \( \nabla \varphi \) around \( \bar{x} \), respectively.
Consider the class of optimization problems

\[
\text{minimize } \varphi(x) := f(x) + g(x), \quad x \in \mathbb{R}^n,
\]

where \( f: \mathbb{R}^n \to \mathbb{R} \) is convex and smooth, while the regularizer \( g: \mathbb{R}^n \to \overline{\mathbb{R}} \) is convex and extended-real-valued. This class encompasses problems of constrained optimization. For each \( \gamma > 0 \) consider the proximal mapping of the regularizer \( g \) by

\[
\text{Prox}_{\gamma g}(x) := \arg\min_{y \in \mathbb{R}^n} \left\{ g(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}
\]

and define [PB13] the forward-backward envelope (FBE) of \( \varphi \)

\[
\varphi_{\gamma}(x) := \inf_{y \in \mathbb{R}^n} \left\{ f(x) + \langle \nabla f(x), y - x \rangle + g(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}.
\]

If \( f \) is \( C^2 \)-smooth with the Lipschitz continuous \( \nabla f \), then

\[
\nabla \varphi_{\gamma}(x) = \gamma^{-1} \left( I - \gamma \nabla^2 f(x) \right) \left( x - \text{Prox}_{\gamma g}(x - \gamma \nabla f(x)) \right).
\]
Algorithm 3 Coderivative-based damped Newton algorithm for convex composite optimization with $f(x) := \frac{1}{2}\langle Ax, x \rangle + \langle b, x \rangle + \alpha$

**Input:** $x^0 \in \mathbb{R}^n$, $\gamma > 0$ such that $B := I - \gamma A \succ 0$, $\sigma \in (0, \frac{1}{2})$, $\beta \in (0, 1)$, and $\varphi_\gamma$ is FBE

1. **for** $k = 0, 1, \ldots$ **do**
2. 
   If $\nabla \varphi_\gamma(x^k) \neq 0$, set $u^k := x^k - \gamma(Ax^k + b)$, $v^k := \text{Prox}_{\gamma g}(u^k)$
3. 
   Find $d^k$ as $-\frac{1}{\gamma}(x^k - v^k) - Ad^k \in \partial^2 g(v^k, \frac{1}{\gamma}(u^k - v^k))(x^k - v^k + d^k)$
4. 
   Set $\tau_k := 1$
5. 
   **while** $\varphi_\gamma(x^k + \tau_k d^k) > \varphi_\gamma(x^k) + \sigma \tau_k \langle \nabla \varphi_\gamma(x^k), d^k \rangle$ **do**
6. 
   set $\tau_k := \beta \tau_k$
7. 
   **end while**
8. 
   Set $x^{k+1} := x^k + \tau_k d^k$
9. **end for**
TWICE EPI-DIFFERENTIABILITY

The second subderivative [RW98] of \( \varphi : \mathbb{R}^n \to \overline{\mathbb{R}} \) at \( \bar{x} \) for \( v, w \) is

\[
d^2 \varphi(\bar{x}, v)(w) := \liminf_{\tau \downarrow 0, u \to w} \Delta^2_T \varphi(\bar{x}, v)(u)
\]

where

\[
\Delta^2_T \varphi(\bar{x}, v)(u) := \frac{\varphi(\bar{x} + \tau u) - \varphi(\bar{x}) - \tau \langle v, u \rangle}{\frac{1}{2} \tau^2}.
\]

The function \( \varphi \) is twice epi-differentiable at \( \bar{x} \) for \( v \) if for every \( w \) and \( \tau_k \downarrow 0 \) there exists a sequence \( w^k \to w \) such that

\[
\frac{\varphi(\bar{x} + \tau_k w^k) - \varphi(\bar{x}) - \tau_k \langle v, w^k \rangle}{\frac{1}{2} \tau_k^2} \to d^2 \varphi(\bar{x}, v)(w).
\]

A general and verifiable condition for twice epi-differentiability is provided by parabolic regularity, which covers a large territory in second-order variational analysis and optimization [MMS21].
THEOREM [KMPT21] If $A$ is positive-definite, then Algorithm 3 generates a sequence $\{x^k\}$ such that it globally R-linearly converges to $\bar{x}$, which a tilt-stable local minimizer of $\varphi$ with modulus $\kappa := 1/\lambda_{\min}(A)$. Furthermore, the convergence rate of $\{x^k\}$ is at least Q-superlinear if $\partial g$ is semismooth* at $(\bar{x}, \bar{v})$, where $\bar{v} := -A\bar{x} - b$, and if either one of two following conditions is satisfied:

1. $\sigma \in (0, 1/(2LK))$, where $L := 2 \left(1 - \gamma \lambda_{\min}(A)\right)/\gamma$ and $K := \kappa + \gamma \|B^{-1}\|$.
2. $g$ is twice epi-differentiable at $\bar{x}$ for $\bar{v}$. 


Algorithm 4  Coderivative-based Levenberg-Marquardt algorithm for convex composite optimization

Input:  \( x^0 \in \mathbb{IR}^n, \gamma > 0 \) such that \( B := I - \gamma A \succ 0, \lambda > 0, \sigma \in \left(0, \frac{1}{2}\right), \beta \in (0, 1), \) and \( \varphi_\gamma \) is FBE

1: for \( k = 0, 1, \ldots \) do
2: Set \( u^k := x^k - \gamma (Ax^k + b), v^k := \text{Prox}_{\gamma g}(u^k), \mu_k := \lambda \|\nabla \varphi_\gamma(x^k)\| \)
3: Set \( d^k := Bz^k, \) where \( z^k \) is from \( -\frac{1}{\gamma}(x^k - v^k) - (\mu_k I + AB)z^k \in \partial^2 g\left(v^k, \frac{1}{\gamma}(u^k - v^k)\right)(x^k - v^k + (B + \gamma \mu_k I)z^k) \)
4: Set \( \tau_k = 1 \)
5: while \( \varphi_\gamma(x^k + \tau_k d^k) > \varphi_\gamma(x^k) + \sigma \tau_k \langle \nabla \varphi_\gamma(x^k), d^k \rangle \) do
6: set \( \tau_k := \beta \tau_k \)
7: end while
8: Set \( x^{k+1} := x^k + \tau_k d^k \)
9: end for
GLOBAL CONVERGENCE OF ALGORITHM 4

**THEOREM [KMPT21]** Let $A$ be positive-semidefinite. Then:

- Any limiting point $\bar{x}$ of iterates $\{x^k\}$ of Algorithm 4 is an optimal solutions to $\varphi$.
- If $\partial \varphi$ is metrically regular at $(\bar{x},0)$ with modulus $\kappa > 0$, then the sequence $\{x^k\}$ globally $R$-linearly converges to $\bar{x}$, and $\bar{x}$ is a tilt-stable local minimizer of $\varphi$ with modulus $\kappa$.
- The rate of convergence of $\{x^k\}$ is at least $Q$-superlinear if $\partial g$ is semismooth* at $(\bar{x},\bar{v})$, where $\bar{v} := -A\bar{x} - b$, and if either one of following two conditions holds:
  - (a) $\sigma \in (0, 1/(2LK))$, where $L := 2 \left(1 - \frac{\gamma \lambda_{\min}(A)}{\gamma}\right)/\gamma$ and $K := \kappa + \gamma\|B^{-1}\|$.
  - (b) $g$ is twice epi-differentiable at $\bar{x}$ for $\bar{v}$.
The basic Lasso problem appeared in statistic [T86] as

$$\minimize \varphi(x) := \frac{1}{2} \| Ax - b \|_2^2 + \mu \| x \|_1, \quad x \in \mathbb{R}^n,$$

where $A$ is an $m \times n$ matrix and $\mu > 0$. All the parameters of Algorithms 3 (GDNM) and Algorithm 4 (GLMM) are computed entirely in terms of given data of the Lasso problem.

Numerical experiments are conducted for GDNM and GLMM by using random data with $\mu := 10^{-3}$ and compare with the performance of ADMM [BPCPE10], FISTA [BT09] and SSNAL [LST18].

The conducted experiments show that both GDNM and GLMM behave better (exhibiting the $Q$-superlinear convergence) than the other algorithms for $m \geq n$. It may not be the case for $m < n$ when GLMM behaves better than GDNM and often better than FISTA and ADMM but usually worse than SSNAL.
# SOLVING LASSO ON RANDOM INSTANCES

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