Vector Optimization w.r.t. Relatively Solid Convex Cones in Real Linear Spaces

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A motivation of studying generalized interiors

It is known that in **vector optimization** in infinite dimensional linear spaces difficulties may arise because of the possible non-solidness of ordering cones (for instance in the fields of optimal control, approximation theory, duality theory). Thus, it is of increasing interest to derive optimality conditions and duality results for such vector optimization problems using **generalized interiority conditions**. Such conditions can be formulated using the well-established generalized interiority notions given by *quasi-interior*, *quasi-relative interior*, *algebraic interior* (also known as *core*), *relative algebraic interior* (also known as *intrinsic core*, *pseudo relative interior* or *intrinsic relative interior*). Moreover, it is known that for defining **Pareto-type solution concepts** of vector optimization problems, generalized interiority notions are also useful.

A motivation of studying generalized algebraic interiors

Having two real linear spaces X and E, a vector-valued objective function $f : X \to E$, a certain set of constraints $\Omega \subseteq X$, a convex (ordering) cone $K \subseteq E$ (with possibly empty algebraic interior), a **vector optimization problem** is defined by

$$\begin{cases} f(x) \to \min \ \text{ w.r.t. } K \\ x \in \Omega. \end{cases}$$

For this problem, a useful solution concept is to say that a point $\bar{x} \in \Omega$ is **optimal** if

$$\{x \in \Omega \mid f(x) \in f(\bar{x}) - \mathrm{icor}\, K\} = \emptyset, \tag{1}$$

where $\mathrm{icor}\, K$ denotes the **intrinsic core** of K.

A motivation of studying generalized algebraic interiors

By involving an appropriate set $S \subseteq E \setminus \{0\}$ with $\mathrm{icor}\, K \subseteq S$ one can define a stronger solution concept by replacing (1) by

$$\{x \in \Omega \mid f(x) \in f(\bar{x}) - S\} = \emptyset. \tag{2}$$

Notice that (2) implies (1). This leads to other solutions concepts such as the well-known concepts of **Pareto efficiency** (i.e., $\bar{x}$ satisfies (2) for $S := K \setminus (-K)$) or **proper Pareto efficiency** (i.e., $\bar{x}$ satisfies (2) with $S := \mathrm{cor}\, C$ for some generalized dilating cone C).

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Convex core topology

Let

$$E' := \{x' : E \to \mathbb{R} \mid x' \text{ is a linear functional}\}$$

be the **algebraic dual space** of E.

Consider the strongest locally convex topology τ_c on E, that is generated by the family of all the semi-norms defined on E; τ_c is known as the **convex core topology**.

The topological dual space of E, namely $(E, \tau_c)^*$, is exactly the algebraic dual space E'.

References: Holmes (1975), Khan, Tammer and Zălinescu (2015), and Zălinescu and Novo (2021)

Some algebraic definitions

Algebraic notions

- $\operatorname{cor} \Omega := \{x \in \Omega \mid \forall\, v \in E \;\exists\, \delta > 0 : \; x + [0, \delta] \cdot v \subseteq \Omega\}$.
- $\operatorname{icor} \Omega := \{x \in \Omega \mid \forall\, v \in \operatorname{aff}(\Omega - \Omega) \;\exists\, \delta > 0 : \; x + [0, \delta] \cdot v \subseteq \Omega\}$.
- $\operatorname{acl} \Omega := \{x \in E \mid \exists\, \overline{x} \in \Omega \;\; [\overline{x}, x) \subseteq \Omega\}$.

As usual, the set $\Omega \subseteq E$ is said to be

- (algebraically) solid if $\operatorname{cor} \Omega \neq \emptyset$.
- relatively (algebraically) solid if $\operatorname{icor} \Omega \neq \emptyset$.
- algebraically closed if $\operatorname{acl} \Omega = \Omega$.

Special properties of convex sets

- For any linear topology τ on E, we have $\operatorname{acl}\Omega \subseteq \operatorname{cl}_\tau \Omega$. Hence, if $\operatorname{cl}_\tau \Omega = \Omega$ then $\Omega = \operatorname{acl}\Omega$.
- For any relatively solid, convex set Ω, we have
$$\operatorname{acl}\Omega = \operatorname{cl}_{\tau_C} \Omega.$$
- For any convex set Ω, we have
$$\operatorname{cor}\Omega = \operatorname{int}_{\tau_c} \Omega \quad \text{and} \quad \operatorname{icor}\Omega = \operatorname{rint}_{\tau_c} \Omega.$$

Algebraic properties of convex cones

Definition 1

Let E be a real linear space. A set $K \subseteq E$ is called a ***convex cone*** *if $0 \in K = \mathbb{R}_+ \cdot K = K + K$.*

Assume that $K \subseteq E$ is a convex cone. Define $\ell(K) := K \cap (-K)$.

Lemma 1

1° $Q := K \setminus \ell(K)$ *is a convex set and* $Q_0 := Q \cup \{0\}$ *is a pointed, convex cone.*

2° *For all* $\overline{x} \in Q$ *and all* $x \in K$ *we have* $[\overline{x}, x) \subseteq Q$.

3° *If* $Q \neq \emptyset$, *then* $Q \subseteq K \subseteq \mathrm{acl}\, Q$.

4° $Q \neq \emptyset \iff \mathrm{icor}\, K \subseteq Q \iff 0 \notin \mathrm{icor}\, K$.

The dual cone of the convex cone K

Consider the following subsets of E'

- $K^+ := \{x' \in E' \mid \forall\, k \in K :\ x'(k) \geq 0\}$,
- $K^\# := \{x' \in E' \mid \forall\, k \in K \setminus \{0\} :\ x'(k) > 0\}$,
- $K^\& := \{x' \in E' \mid \forall\, k \in K \setminus \ell(K) :\ x'(k) > 0\}$.

New algebraic properties of dual cones of convex cones

Theorem 2

Assume that K is a τ_c-closed, convex cone. Then, the following hold:

1° $\operatorname{cor} K^+ \subseteq K^\#$.

2° $\operatorname{icor} K^+ \subseteq K^\&$.

Now, assume that E has finite dimension. Then, we have:

3° *If K^+ is solid, then $\operatorname{cor} K^+ = K^\#$.*

4° $\operatorname{icor} K^+ = K^+ \cap K^\&$.

5° *If either $K \neq \ell(K)$ or $K = \{0\}$, then $\operatorname{icor} K^+ = K^\&$.*

Separation theorems in linear spaces using algebraic notions

Proposition 3 (Holmes' support theorem)

Assume that $\Omega \subseteq E$ is a relatively solid, convex set, and $x \in E$. Then, the following assertions are equivalent:

1° $x \notin \operatorname{icor} \Omega$.

2° $\exists\, x' \in E', \alpha \in \mathbb{R}, \forall\, \omega \in \operatorname{icor} \Omega : \; x'(x) \leq \alpha < x'(\omega)$.

Corollary 4

Assume that $\Omega^1, \Omega^2 \subseteq E$ are relatively solid, convex sets. Then, the following assertions are equivalent:

1° $(\operatorname{icor} \Omega^1) \cap (\operatorname{icor} \Omega^2) = \emptyset$.

2° $\exists\, x' \in E', \alpha \in \mathbb{R}, \forall\, \omega^1 \in \operatorname{icor} \Omega^1, \omega^2 \in \operatorname{icor} \Omega^2 :$
$0 \leq \alpha < x'(\omega^1) - x'(\omega^2)$.

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Vector optimization problems

Assume that X and E are real linear spaces, and E is pre-ordered by a convex cone K. In **vector optimization**, one aims to minimize a vector-valued function

$$f : X \to E$$

over a nonempty set Ω of X, i.e.,

$$\begin{cases} f(x) \to \min \ \text{ w.r.t. } K \\ x \in \Omega. \end{cases} \tag{$\mathcal{P}_\Omega$}$$

Typically one is looking for so-called **Pareto efficient solutions** of the vector optimization problem $(\mathcal{P}_\Omega)$. Notice, the ideas for such solution concepts for vector optimization problems date back to **Edgeworth** (1881) and **Pareto** (1906).

Binary relations defined on E

We assume now that

$$K \subseteq E \text{ is a convex cone with } K \neq \ell(K). \tag{3}$$

It is well-known that K induces on E a preorder relation $\leqq_K$ defined, for any two points $y, \overline{y} \in E$, by

$$y \leqq_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - K.$$

Moreover, consider the following binary relations:

$$y \leq_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - K \setminus \ell(K),$$
$$y <_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - \mathrm{icor}\, K.$$

Pareto efficiency in vector optimization

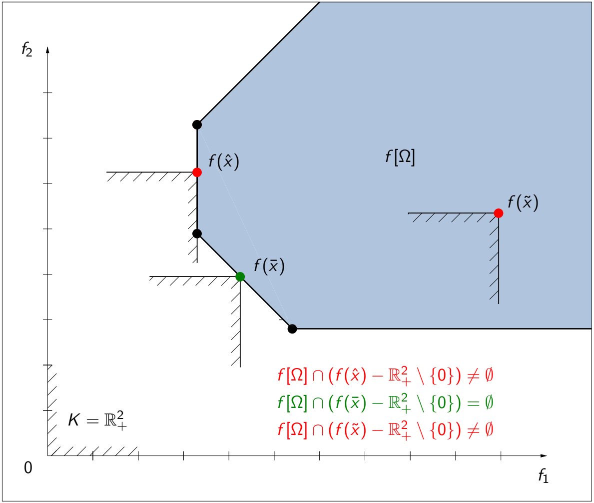
Definition 2 (Pareto efficiency)

A point $\overline{x} \in \Omega$ is said to be a ***Pareto efficient solution*** *if for any $x \in \Omega$ the condition $f(x) \leqq_K f(\overline{x})$ implies $f(\overline{x}) \leqq_K f(x)$.*

The set of all Pareto efficient solutions of $(\mathcal{P}_\Omega)$ is denoted by

$$\mathrm{Eff}(\Omega \mid f, K) := \{\overline{x} \in \Omega \mid \forall x \in \Omega : \; f(x) \leqq_K f(\overline{x}) \; \Rightarrow \; f(\overline{x}) \leqq_K f(x)\}.$$

Example: Pareto efficiency in vector optimization



Weak Pareto efficiency in vector optimization

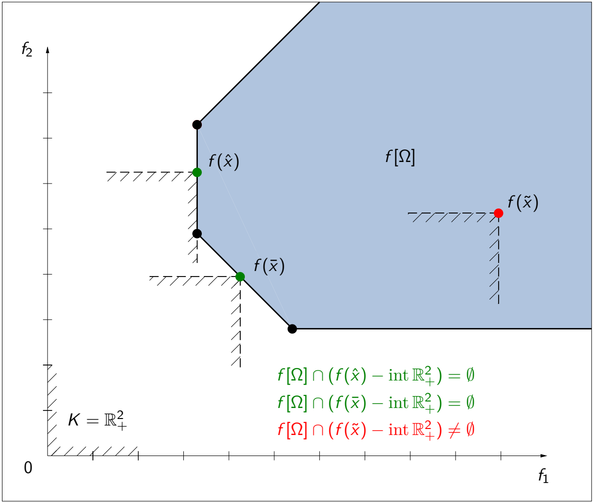
Definition 3 (Weak Pareto efficiency)

*A point $\overline{x} \in \Omega$ is said to be a **weakly Pareto efficient solution** if there is no $x \in \Omega$ such that $f(x) <_K f(\overline{x})$.*

The set of all weakly Pareto efficient solutions of $(\mathcal{P}_\Omega)$ is denoted by

$$\mathrm{WEff}(\Omega \mid f, K) := \{\overline{x} \in \Omega \mid \nexists x \in \Omega : \ f(x) <_K f(\overline{x})\}.$$

Example: Weak Pareto efficiency in vector optimization



Generalized dilating cones

As usual for Henig-type proper efficiency concepts, **(generalized) dilating cones** for the cone K (which satisfies (3)) will play an important role. Our considered proper efficiency concepts will mainly be based on two specific families of convex cones, namely $\mathcal{C}(K)$ and $\mathcal{D}(K)$, that we introduce below:

$$\mathcal{C}(K) := \{C \subseteq E \mid C \text{ is a convex cone with } K \setminus \ell(K) \subseteq \mathrm{icor}\, C \text{ and } C \neq \ell(C)\}$$

and

$$\mathcal{D}(K) := \{D \subseteq E \mid D \text{ is a nontrivial, convex cone with } K \setminus \ell(K) \subseteq \mathrm{cor}\, D\}.$$

Henig proper efficiency

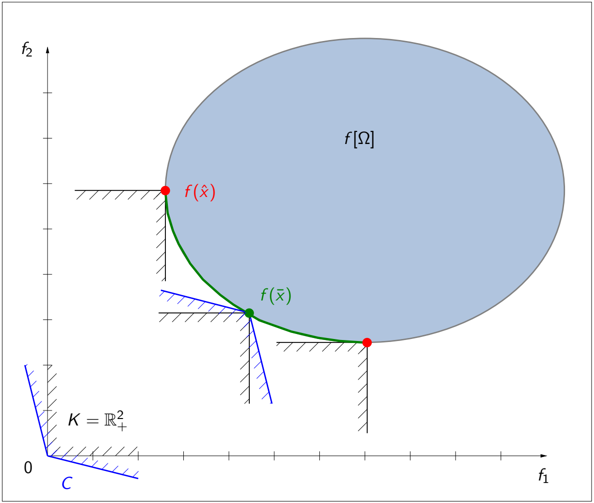
Definition 4 (Proper efficiency in the sense of Henig)

*A point $x \in \Omega$ is said to be a **classical Henig properly efficient solution** if there is a nontrivial, convex cone $D \subseteq E$ with $K \setminus \ell(K) \subseteq \operatorname{cor} D$ (i.e., $D \in \mathcal{D}(K)$) such that $x \in \mathrm{Eff}(\Omega \mid f, D)$.*

The set of all classical Henig properly efficient solutions of $(\mathcal{P}_\Omega)$ is denoted by

$$\mathrm{PEff}_c(\Omega \mid f, K).$$

Example: Henig proper efficiency in vector optimization



Henig-type proper efficiency

Definition 5 (Extended proper efficiency in the sense of Henig)

A point $x \in \Omega$ is said to be a ***Henig properly efficient solution*** *if there is a convex cone $C \subseteq E$ with $K \setminus \ell(K) \subseteq \operatorname{icor} C$ and $C \neq \ell(C)$ (i.e., $C \in \mathcal{C}(K)$) such that $x \in \operatorname{Eff}(\Omega \mid f, C)$.*

The set of all Henig properly efficient solutions of $(\mathcal{P}_\Omega)$ is denoted by

$$\operatorname{PEff}(\Omega \mid f, K).$$

Relationships between the solution concepts

Lemma 5

Suppose that K satisfies (3). Then, the following assertions hold:

1° $\mathrm{PEff}_c(\Omega \mid f, K) \subseteq \mathrm{PEff}(\Omega \mid f, K) \subseteq \mathrm{Eff}(\Omega \mid f, K) \subseteq \mathrm{WEff}(\Omega \mid f, K)$.

2° *If $\mathcal{C}(K) = \mathcal{D}(K)$ (e.g., if K is solid), then* $\mathrm{PEff}_c(\Omega \mid f, K) = \mathrm{PEff}(\Omega \mid f, K)$.

3° *If $\mathcal{C}(K) = \emptyset$ ($\iff \mathcal{D}(K) = \emptyset \iff K^{\&} = \emptyset$), then* $\mathrm{PEff}_c(\Omega \mid f, K) = \mathrm{PEff}(\Omega \mid f, K) = \emptyset$.

4° $\mathrm{PEff}_c(\Omega \mid f, K) = \bigcup_{D \in \mathcal{D}(K)} \mathrm{Eff}(\Omega \mid f, D) = \bigcup_{D \in \mathcal{D}(K)} \mathrm{WEff}(\Omega \mid f, D)$.

5° $\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{C \in \mathcal{C}(K)} \mathrm{Eff}(\Omega \mid f, C) = \bigcup_{C \in \mathcal{C}(K)} \mathrm{WEff}(\Omega \mid f, C)$.

It is possible that

$$\mathrm{PEff}_c(\Omega \mid f, K) \subsetneq \mathrm{PEff}(\Omega \mid f, K) \subsetneq \mathrm{Eff}(\Omega \mid f, K).$$

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Monotonicity concepts for real-valued functions

Definition 6

*Given a binary relation $\sim_E \in \{\leqq_K, \leq_K, <_K\}$, a function $\varphi : E \to \mathbb{R}$ is said to be $\sim_E$-**increasing** if*

$$\forall\, y, \overline{y} \in E :\ y \sim_E \overline{y} \ \Rightarrow\ \varphi(y) < \varphi(\overline{y}).$$

Recall that

$$\begin{aligned} y \leqq_K \overline{y} \quad &:\Longleftrightarrow \quad y \in \overline{y} - K, \\ y \leq_K \overline{y} \quad &:\Longleftrightarrow \quad y \in \overline{y} - K \setminus \ell(K), \\ y <_K \overline{y} \quad &:\Longleftrightarrow \quad y \in \overline{y} - \operatorname{icor} K. \end{aligned}$$

Scalarization results

Lemma 6

Consider a real-valued function $\varphi : E \to \mathbb{R}$*. Then, the following assertions hold:*

1° *If* φ *is* $<_K$*-increasing, then* $\operatorname{argmin}_{x\in\Omega} (\varphi \circ f)(x) \subseteq \mathrm{WEff}(\Omega \mid f, K)$.

2° *If* φ *is* $\leq_K$*-increasing, then* $\operatorname{argmin}_{x\in\Omega} (\varphi \circ f)(x) \subseteq \mathrm{Eff}(\Omega \mid f, K)$.

3° *If* φ *is* $<_D$*-increasing for some* $D \in \mathcal{D}(K)$*, then* $\operatorname{argmin}_{x\in\Omega} (\varphi \circ f)(x) \subseteq \mathrm{PEff}_c(\Omega \mid f, K)$.

4° *If* φ *is* $<_C$*-increasing for some* $C \in \mathcal{C}(K)$*, then* $\operatorname{argmin}_{x\in\Omega} (\varphi \circ f)(x) \subseteq \mathrm{PEff}(\Omega \mid f, K)$.

Linear scalarization results

Lemma 7

Suppose that K satisfies (3). Then:

1° *For any $x' \in K^+ \setminus \ell(K^+)$, we have* $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \mathrm{WEff}(\Omega \mid f, K)$.

2° *For any $x' \in K^{\&}$, we have* $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \mathrm{PEff}_c(\Omega \mid f, K) \subseteq \mathrm{PEff}(\Omega \mid f, K)$.

K-convexlike functions

As usual, the vector function $f : X \to E$ is called

- K-**convex** on the convex set $\Omega \subseteq X$ if, for any $x, \bar{x} \in \Omega$ and $\lambda \in (0,1)$, we have
  $f(\lambda x + (1-\lambda)\bar{x}) \in \lambda f(x) + (1-\lambda) f(\bar{x}) - K$.
- K-**convexlike** on $\Omega \subseteq X$ if $f[\Omega] + K$ is a convex set.

Remark 8

Any K-convex function f is K-convexlike as well.

Scalarization results (weak Pareto efficiency)

Theorem 9

Suppose that K is relatively solid and satisfies (3). In addition, assume that the function f is K-convexlike on Ω, and $f[\Omega] + K$ is relatively solid. Then, the following assertions hold:

1°
$$\mathrm{WEff}(\Omega \mid f, K) \subseteq \bigcup_{x' \in K^+ \setminus \{0\}} \operatorname{argmin}_{x \in \Omega} (x' \circ f)(x).$$

2° *If K^+ is pointed, then*
$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \{0\}} \operatorname{argmin}_{x \in \Omega} (x' \circ f)(x).$$

Scalarization results (weak Pareto efficiency)

Theorem 9 (part 2)

3° If $\bar{x} \in \mathrm{WEff}(\Omega \mid f, K)$ and $f(\bar{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then

$$\bar{x} \in \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

4° If $f[\mathrm{WEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

Scalarization result (Henig proper efficiency)

Theorem 10

Suppose that K satisfies (3). Assume that the function f is K-convexlike on Ω. Then:

1° $$\mathrm{PEff}_c(\Omega \mid f, K) = \bigcup_{x' \in K^{\&}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

2° *If K is τ_c-closed, and E has finite dimension, then*

$$\mathrm{PEff}_c(\Omega \mid f, K) = \bigcup_{x' \in \mathrm{icor}\, K^{+}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

Scalarization result (Henig-type proper efficiency)

Theorem 11

Suppose that K is relatively solid and satisfies (3). In addition, assume that the function f is K-convexlike on Ω. Then:

1° *If* $\bar{x} \in \mathrm{PEff}(\Omega \mid f, K)$ *and* $f(\bar{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, *then*

$$\bar{x} \in \bigcup_{x' \in K^{\&}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

2° *If* $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, *then*

$$\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{x' \in K^{\&}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

Scalarization results (Henig-type proper efficiency)

Theorem 11 (part 2)

3° If K is τ_c-closed, E has finite dimension, and $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then

$$\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{x' \in \mathrm{icor}\, K^+} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

Choice of references

- R. B. Holmes, **Geometric Functional Analysis and its Applications**, Springer-Verlag, New York, 1975.
- C. Zălinescu, **Convex Analysis in General Vector Spaces**. World Scientific Publishing, River Edge, 2002.
- J. Jahn, **Vector Optimization - Theory, Applications, and Extensions**, Springer-Verlag, Berlin, Heidelberg, 2011.
- A. A. Khan, C. Tammer and C. Zălinescu, **Set-valued Optimization: An Introduction with Applications**. Springer-Verlag, Berlin, Heidelberg, 2015.
- B. Khazayel, A. Farajzadeh, C. Günther, C. Tammer, **On the intrinsic core of convex cones in real linear spaces**, SIAM Journal on Optimization, 31(2) (2021), pp. 1276–1298. https://doi.org/10.1137/19M1283148
- C. Günther, B. Khazayel, and C. Tammer, **Vector Optimization w.r.t. Relatively Solid Convex Cones in Real Linear Spaces**, Report, Martin Luther University Halle-Wittenberg, 04/2021 (submitted) http://www.optimization-online.org/DB_HTML/2019/08/7349.html

Choice of references

- M. Adán and V. Novo, **Proper efficiency in vector optimization on real linear spaces**, J. Optim. Theory Appl., 121 (2004), pp. 515–540.
- M. Adán and V. Novo, **Weak efficiency in vector optimization using a closure of algebraic type under cone-convexlikeness**, European J. Oper. Res., 149 (2003), pp. 641–653.
- T. Q. Bao and B. S. Mordukhovich, **Relative Pareto minimizers for multiobjective problems: existence and optimality conditions**. Math. Program. 122 (2010), pp. 301–347.
- J. M. Borwein and R. Goebel, **Notions of relative interior in Banach spaces**. J. Math. Sci. 115(4) (2003), pp. 2542–2553.
- V. Novo and C. Zălinescu, **On relatively solid convex cones in real linear spaces**, J. Optim. Theory Appl., 188 (2021), pp. 277–290.
- N. Popovici, **Explicitly quasiconvex set-valued optimization**. Glob. Optim. 38 (2007), pp. 103–118.
- C. Zălinescu, **On the use of the quasi-relative interior in optimization**, Optimization, 64 (2015), pp. 1795–1823.

Thank you for your attention!