Warped Proximal Iterations

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NC STATE UNIVERSITY

Lev Bregman 1965's paper

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МАТЕМАТИКА

Л. М. БРЭГМАН

НАХОЖДЕНИЕ ОБЩЕЙ ТОЧКИ ВЫПУКЛЫХ МНОЖЕСТВ МЕТОДОМ ПОСЛЕДОВАТЕЛЬНОГО ПРОЕКТИРОВАНИЯ

(Представлено академиком Л. В. Канторовичем 7 XII 1964)

В настоящей статье рассматривается итеративный метод для нахождения общей точки выпуклых множеств. Этот метод может быть применен для задач оптимального программирования и для некоторых других.

Пусть в действительном гильбертовом пространстве H с расстоянием ρ заданы замкнутые выпуклые множества A_i, $i \in I$, где I — некоторое множество индексов. Пусть $R = \bigcap_{i \in I} A_i$ не пусто. Требуется найти какую-либо точку $x \in R$. Рассмотрим следующий итеративный процесс: берем любую точку $x_0 \in H$, затем выбираем $i(x_0) \in I$ и в множестве $A_{i(x_0)}$ находим точку x_1, ближайшую к x_0, затем так же выбираем $i(x_1) \in I$ и в множестве $A_{i(x_1)}$ находим точку x_2, ближайшую к x_1, и т. д.

PART 1:

Background

Monotone operator splitting

- Basic problem: $\mathcal{X}$ a real Banach space. Given a maximally monotone operator $M \colon \mathcal{X} \to 2^{\mathcal{X}^*}$, find $x \in \mathcal{X}$ such that $0 \in Mx$.

Monotone operator splitting

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- Considerable range of applications: optimization,
  - Subdifferential: $M = \partial f$ (Fermat's rule)
  - Kuhn-Tucker operator: $M = \begin{bmatrix} \partial f & L^* \\ -L & \partial g^* \end{bmatrix}$.

    (Rockafellar 1967)
  - etc. (Eckstein 1994, PLC 2018, Bùi/PLC 2020).

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- Considerable range of applications: optimization, variational inequalities, statistics, mechanics, neural networks, finance, partial differential equations, optimal transportation, signal and image processing, control, game theory, machine learning, economics, mean fields games, etc.

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- Basic problem: $\mathcal{X}$ a real Banach space. Given a maximally monotone operator $M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$, find $x \in \mathcal{X}$ such that $0 \in Mx$.
- The proximal point algorithm (Bellman 1966, Martinet 1970, Rockafellar 1976):

$$x_{n+1} = J_M x_n, \text{ where } J_M = (\mathrm{Id} + M)^{-1} \text{ is the resolvent of } M.$$

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- Acknowledging the fact that J_M may be hard to implement, *splitting methods* have been developed: the goal is to express M as a combination of operators, and devise an algorithm that uses these operators individually.

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- *Splitting methods*: express M as a combination of operators, and devise an algorithm that uses these operators individually.
- The following structures have been considered:

$$M = A + B$$

(Mercier 1979, Lions/Mercier 1979, Tseng 2000)

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$$M = \sum_{k=1}^{p} A_k$$

(Spingarn 1983, Gol'stein 1985, Eckstein/Svaiter 2009, PLC 2009)

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$$M = \sum_{k=1}^{p} L_k^* \circ B_k \circ L_k$$

(Briceño-Arias/PLC 2011)

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- *Splitting methods*: express M as a combination of operators, and devise an algorithm that uses these operators individually.
- The following structures have been considered:

$$M = A + \sum_{k=1}^{p} L_k^* \circ (B_k \,\square\, D_k) \circ L_k + C$$

(PLC/Pesquet 2012, Vũ 2013, Condat 2013, Boţ/Hendrich 2013)

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(Raguet/Fadili/Peyré 2013, Briceño-Arias 2015, Davis/Yin 2017, Raguet 2019)

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- *Splitting methods*: express M as a combination of operators, and devise an algorithm that uses these operators individually.
- The following structures have been considered:

$$M\colon (x_1,\ldots,x_m) \mapsto \bigtimes_{i=1}^{m} \Bigg(A_i x_i + C_i x_i + Q_i x_i + \sum_{k=1}^{p} L_{ki}^{*} \Bigg(\Big((B_k^m + B_k^c + B_k^l) \,\square\, (D_k^m + D_k^c + D_k^l) \Big) \Bigg(\sum_{j=1}^{m} L_{kj} x_j \Bigg) \Bigg) \Bigg)$$

(Bùi/PLC 2021)

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- *Splitting methods*: express M as a combination of operators, and devise an algorithm that uses these operators individually.
- ... which models in particular

$$\underset{x_1 \in \mathcal{X}_1, \ldots, x_m \in \mathcal{X}_m}{\text{minimize}} \quad \sum_{i=1}^{m} \big(f_i(x_i) + \varphi_i(x_i)\big) + \sum_{k=1}^{p} \big((g_k + \psi_k) \,\square\, h_k\big)\Bigg(\sum_{j \in I} L_{kj} x_j\Bigg).$$

(Bùi/PLC 2021)

- Can we provide a synthetic view of some of these methods in terms of a resolvent iteration akin to the proximal point algorithm?

PART 2:

The warped resolvent

The warped resolvent: Definition

- $\mathcal{X}$ is a reflexive real Banach space with topological dual $\mathcal{X}^*$.
- An operator $M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ is *monotone* if

$$\big(\forall (x_1,x_1^*) \in \operatorname{gra} M\big)\big(\forall (x_2,x_2^*) \in \operatorname{gra} M\big) \quad \langle x_1 - x_2, x_1^* - x_2^* \rangle \geqslant 0,$$

and *maximally monotone* if, furthermore, no point in $\mathcal{X} \times \mathcal{X}^*$ can be added to $\operatorname{gra} M$ without compromising monotonicity.

Definition

Let $\varnothing \neq D \subset \mathcal{X}$, let $K\colon D \to \mathcal{X}^*$, and let $M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ be such that $\operatorname{ran} K \subset \operatorname{ran}(K+M)$ and $K+M$ is injective. [1] The warped resolvent of M with kernel K is $J_M^K = (K+M)^{-1} \circ K$.

[1] $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ is injective if $(\forall x \in \mathcal{X})(\forall y \in \mathcal{X})\; Ax \cap Ay \neq \varnothing \Rightarrow x = y$.

The warped resolvent: Examples

$M\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ is maximally monotone.

- If $\mathcal{X}$ is Hilbertian and $K = \mathrm{Id}$, J_M^K is the classical resolvent.

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- If $K = \nabla f$ and $M = N_C$, J_M^K is the Bregman projection operator (1967).
- If $\mathcal{X}$ is strictly convex with normalized duality mapping K, then J_M^K is the extended resolvent of (Kassay, 1985).

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- If $\mathcal{X}$ is strictly convex with normalized duality mapping K, then J_M^K is the extended resolvent of (Kassay, 1985).
- Let $f\colon \mathcal{X} \to \left]-\infty, +\infty\right]$ be a Legendre function such that $\operatorname{dom} M \subset \operatorname{int} \operatorname{dom} f$, and set $K = \nabla f$. Then J_M^K is the D-resolvent of (Bauschke/Borwein/PLC, 2003).

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- $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ and $B\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ are maximally monotone, and $f\colon \mathcal{X} \to \left]-\infty, +\infty\right]$ is a suitable convex function. Set

$$M = A + B \quad \text{and} \quad K\colon \operatorname{int}\operatorname{dom} f \to \mathcal{X}^*\colon x \mapsto \nabla f(x) - Bx.$$

Then $J_M^K = (\nabla f + A)^{-1} \circ (\nabla f - B)$ is the Bregman forward-backward operator to be discussed in Part 4.

The warped resolvent: Examples

- Let $K\colon \mathcal{X} \to \mathcal{X}^*$ be strictly monotone, 3^* monotone, and surjective. Then J_M^K is the K-resolvent of (Bauschke/Wang/Yao, 2010).

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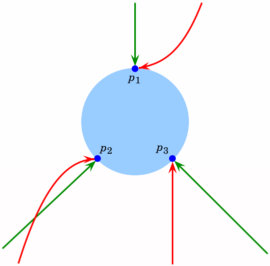
- Let $K\colon \mathcal{X} \to \mathcal{X}^*$ be strictly monotone, 3^* monotone, and surjective. Then J_M^K is the K-resolvent of (Bauschke/Wang/Yao, 2010).
- Let $\varnothing \neq C \subset \mathcal{X}$ be closed and convex, with normal cone operator N_C. The warped projection operator is $\text{proj}_C^K = J_{N_C}^K = (K+N_C)^{-1} \circ K$.



Left: Warped projections onto $B(0;1)$. Sets of points projecting onto p_1, p_2, and p_3 for $K_1 = \mathrm{Id}$ and

$$K_2\colon (\xi_1, \xi_2) \mapsto \left(\frac{\xi_1^3}{2} + \frac{\xi_1}{5} - \xi_2, \xi_1 + \xi_2 \right)$$

Note that K_2 is not a gradient, so this is not a Bregman projector.

The warped resolvent: Properties

- Sufficient conditions for $\operatorname{ran} K \subset \operatorname{ran}(K+M)$ and $K+M$ is injective are given in (Bùi/PLC, 2020).
- $J_M^K \colon D \to D$.
- $\operatorname{Fix} J_M^K = D \cap \operatorname{zer} M$.
- $p = J_M^K x \Leftrightarrow (p, Kx - Kp) \in \operatorname{gra} M$.
- Suppose that M is monotone. Let $x \in D$, and set $y = J_M^K x$ and $y^* = Kx - Ky$. Then

$$\operatorname{zer} M \subset \big\{ z \in \mathcal{X} \mid \langle z - y, y^* \rangle \leqslant 0 \big\}.$$

- Suppose that M is monotone. Set $p = J_M^K x$ and $q = J_M^K y$. Then

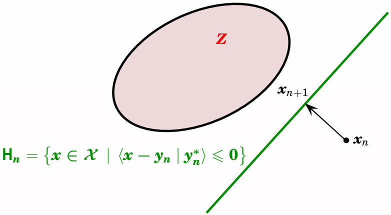
$$\langle p - q, Kx - Ky \rangle \geqslant \langle p - q, Kp - Kq \rangle.$$

PART 3:

Warped proximal iterations in Hilbert spaces

Finding zeros of monotone operators: Geometry

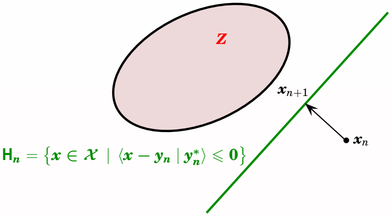
- M maximally monotone with $Z = \operatorname{zer} M \neq \varnothing$.



Finding zeros of monotone operators: Geometry

- M maximally monotone with $Z = \operatorname{zer} M \neq \varnothing$.
- Iterate

$$
\left\lfloor
\begin{array}{l}
(y_n, y_n^*) \in \operatorname{gra} M \\
\lambda_n \in [\varepsilon, 2-\varepsilon] \\
\text{if } \langle y_n - x_n \mid y_n^* \rangle < 0 \\
\left\lfloor \; x_{n+1} = x_n + \lambda_n \langle y_n - x_n \mid y_n^* \rangle y_n^* / \|y_n^*\|^2 \right. \\
\text{else} \\
\left\lfloor \; x_{n+1} = x_n . \right.
\end{array}
\right.
$$



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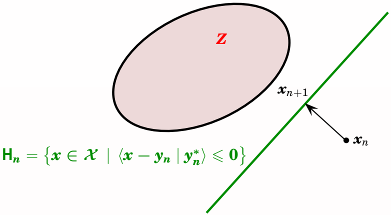

- Weak convergence to a point in Z if weak cluster points are in Z.
- The weak–to–strong convergence principle (Bauschke/PLC, 2001) gives strong convergence of a 2 half-spaces variant.
- How to choose $(y_n, y_n^*) \in \operatorname{gra} M$?

Finding zeros of monotone operators: Geometry

- M maximally monotone with $Z = \operatorname{zer} M \neq \varnothing$.
- Iterate

$$
\left\lfloor
\begin{array}{l}
y_n = J^{K_n}_{\gamma_n M}\widetilde{x}_n \\
y_n^* = \gamma_n^{-1}(K_n\widetilde{x}_n - K_n y_n) \\
\lambda_n \in [\varepsilon, 2-\varepsilon] \\
\text{if } \langle y_n - x_n \mid y_n^* \rangle < 0 \\
\left\lfloor\; x_{n+1} = x_n + \lambda_n \langle y_n - x_n \mid y_n^* \rangle y_n^* / \|y_n^*\|^2 \right. \\
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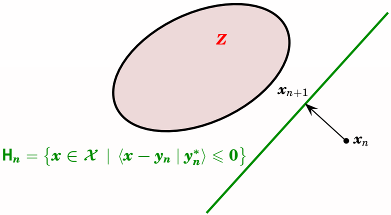

- **Key:** Move beyond Minty's parametrization of $\operatorname{gra} M$ and use a warped resolvent to pick $(y_n, y_n^*) \in \operatorname{gra} M$.
- Simply evaluate a warped resolvent at some point $\widetilde{x}_n$.

Convergence

Notation: $(y^*)^\sharp = y^*/\|y^*\|$ if $y^* \neq 0$; $= 0$ otherwise.

Theorem

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, +\infty[$. For every $n \in \mathbb{N}$, let $\widetilde{x}_n \in \mathcal{X}$ and let $K_n \colon \mathcal{X} \to \mathcal{X}$ be a monotone operator such that ran $K_n \subset$ ran $(K_n + \gamma_n M)$ and $K_n + \gamma_n M$ is injective. Suppose that:

- $\widetilde{x}_n - x_n \to 0$.
- $\langle \widetilde{x}_n - y_n \mid (K_n\widetilde{x}_n - K_n y_n)^\sharp \rangle \to 0 \quad \Rightarrow \quad \begin{cases} \widetilde{x}_n - y_n \rightharpoonup 0 \\ K_n\widetilde{x}_n - K_n y_n \to 0. \end{cases}$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in Z.

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Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in Z.

- We also have a strongly convergent version.

Choosing the evaluation points $(\widetilde{x}_n)_{n\in\mathbb{N}}$

The auxiliary sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ can serve several purposes:

- $\widetilde{x}_n$ can model an additive perturbation of x_n, say $\widetilde{x}_n = x_n + e_n$, where we require only $\|e_n\| \to 0$.

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- Modeling inertia: let $(\alpha_n)_{n\in\mathbb{N}}$ be **any** bounded sequence in $\mathbb{R}$ and set $\widetilde{x}_n = x_n + \alpha_n(x_n - x_{n-1})$.

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- More generally,

$$(\forall n \in \mathbb{N}) \quad \widetilde{x}_n = \sum_{j=0}^{n} \mu_{n,j} x_j.$$

with $\sum_{j=0}^{n} \mu_{n,j} = 1$ and $(1-\mu_{n,n})x_n - \sum_{j=0}^{n-1} \mu_{n,j} x_j \to 0$.

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  with $\sum_{j=0}^{n} \mu_{n,j} = 1$ and $(1-\mu_{n,n})x_n - \sum_{j=0}^{n-1} \mu_{n,j} x_j \to 0$.
- Nonlinear perturbations can also be considered. For instance, at iteration n, $\widetilde{x}_n = \operatorname{proj}_{C_n} x_n$ is an approximation to x_n from some suitable closed convex set $C_n \subset \mathcal{X}$.

Corollary 1

Corollary

Let $A\colon \mathcal{X} \to 2^{\mathcal{X}}$ *be maximally monotone, and let* $B\colon \mathcal{X} \to \mathcal{X}$ *be monotone and* β*-Lipschitzian, with* $\operatorname{zer}(A+B) \neq \varnothing$*. Let* $W_n\colon \mathcal{X} \to \mathcal{X}$ *be* α*-strongly monotone and* χ*-Lipschitzian, and let* $\gamma_n \in [\varepsilon, (\alpha-\varepsilon)/\beta]$*, let* $\lambda_n \in [\varepsilon, 2-\varepsilon]$*, and let* $\mathcal{X} \ni e_n \to 0$*. Furthermore, let* $m > 0$ *and let* $(\mu_{n,j})_{n\in\mathbb{N}, 0\leqslant j\leqslant n}$ *be bounded and satisfy*

- *For every* $n > m$ *and every integer* $j \in [0, n-m-1]$*,* $\mu_{n,j} = 0$*.*
- *For every* $n \in \mathbb{N}$*,* $\sum_{j=0}^{n} \mu_{n,j} = 1$*.*

Iterate

$$
\left\lfloor
\begin{array}{l}
\widetilde{x}_n = e_n + \sum_{j=0}^{n} \mu_{n,j} x_j \\
v_n^* = W_n \widetilde{x}_n - \gamma_n B \widetilde{x}_n \\
y_n = (W_n + \gamma_n A)^{-1} v_n^* \\
y_n^* = \gamma_n^{-1}(v_n^* - W_n y_n) + B y_n \\
\text{if } \langle y_n - x_n \mid y_n^* \rangle < 0 \\
\left\lfloor \; x_{n+1} = x_n + \dfrac{\lambda_n \langle y_n - x_n \mid y_n^* \rangle}{\|y_n^*\|^2} y_n^* \right. \\
\text{else } x_{n+1} = x_n.
\end{array}
\right.
$$

Then $(x_n)_{n\in\mathbb{N}}$ *converges weakly to a point in* $\operatorname{zer}(A+B)$*.*

Proof: $M = A + B$ and $K_n = W_n - \gamma_n B$. **Special case:** Tseng's algorithm.

Corollary 2: Multivariate inclusions

- **Problem:** find $(x_i)_{i\in I} \in \bigtimes_{i\in I} \mathcal{X}_i$ such that

$$(\forall i \in I) \quad 0 \in A_i x_i + \sum_{j\in J} L_{ji}^* \Big((B_j + D_j) \Big(\sum_{k\in I} L_{jk} x_k \Big) \Big) + C_i x_i$$

- **Warping:** Apply the Theorem to

$$M\colon \; \big((x_i)_{i\in I}, (y_j)_{j\in J}, (v_j^*)_{j\in J}\big) \mapsto \Bigg(\bigtimes_{i\in I} \Big(A_i x_i + C_i x_i + \sum_{j\in J} L_{ji}^* v_j^* \Big),$$
$$\bigtimes_{j\in J} (B_j y_j + D_j y_j - v_j^*), \bigtimes_{j\in J} \Big\{ y_j - \sum_{i\in I} L_{ji} x_i \Big\} \Bigg)$$

and $K_n\colon (x, y, v^*) \mapsto$

$$\Bigg(\Big(\gamma_{i,n}^{-1} F_{i,n} x_i - C_i x_i - \sum_{j\in J} L_{ji}^* v_j^* \Big)_{i\in I}, \big(\tau_{j,n}^{-1} W_{j,n} y_j - D_j y_j + v_j^*\big)_{j\in J},$$
$$\Big(-y_j + v_j^* + \sum_{i\in I} L_{ji} x_i \Big)_{j\in J} \Bigg),$$

where $F_{i,n}$ and $W_{j,n}$ are strongly monotone and Lipschitzian.

Corollary 2: Multivariate inclusions

for $n = 0, 1, \ldots$

$$
\left\lfloor
\begin{array}{l}
\text{for every } i \in I \\
\left\lfloor
\begin{array}{l}
l_{i,n}^* = F_{i,n}\widetilde{x}_{i,n} - \gamma_{i,n}C_i\widetilde{x}_{i,n} - \gamma_{i,n}\sum_{j\in J} L_{ji}^*\widetilde{v}_{j,n}^* \\
a_{i,n} = \big(F_{i,n} + \gamma_{i,n}A_i\big)^{-1}(l_{i,n}^* + \gamma_{i,n}s_i^*) \\
o_{i,n}^* = \gamma_{i,n}^{-1}(l_{i,n}^* - F_{i,n}a_{i,n}) + C_ia_{i,n}
\end{array}
\right. \\
\text{for every } j \in J \\
\left\lfloor
\begin{array}{l}
t_{j,n}^* = W_{j,n}\widetilde{y}_{j,n} - \tau_{j,n}D_j\widetilde{y}_{j,n} + \tau_{j,n}\widetilde{v}_{j,n}^* \\
b_{j,n} = \big(W_{j,n} + \tau_{j,n}B_j\big)^{-1}t_{j,n}^* \\
f_{j,n}^* = \tau_{j,n}^{-1}(t_{j,n}^* - W_{j,n}b_{j,n}) + D_jb_{j,n} \\
c_{j,n} = \sum_{i\in I} L_{ji}\widetilde{x}_{i,n} - \widetilde{y}_{j,n} + \widetilde{v}_{j,n}^* - r_j
\end{array}
\right. \\
\text{for every } i \in I \\
\left\lfloor\; a_{i,n}^* = o_{i,n}^* + \sum_{j\in J} L_{ji}^*c_{j,n} \right. \\
\text{for every } j \in J \\
\left\lfloor
\begin{array}{l}
b_{j,n}^* = f_{j,n}^* - c_{j,n} \\
c_{j,n}^* = r_j + b_{j,n} - \sum_{i\in I} L_{ji}a_{i,n}
\end{array}
\right. \\
\sigma_n = \sum_{i\in I}\|a_{i,n}^*\|^2 + \sum_{j\in J}\big(\|b_{j,n}^*\|^2 + \|c_{j,n}^*\|^2\big) \\
\theta_n = \sum_{i\in I}\langle a_{i,n} - x_{i,n} \mid a_{i,n}^*\rangle + \sum_{j\in J}\big(\langle b_{j,n} - y_{j,n} \mid b_{j,n}^*\rangle + \langle c_{j,n} - v_{j,n}^* \mid c_{j,n}^*\rangle\big) \\
\text{if } \theta_n < 0 \\
\left\lfloor\; \rho_n = \lambda_n\theta_n/\sigma_n \right. \\
\text{else} \\
\left\lfloor\; \rho_n = 0 \right. \\
\text{for every } i \in I \\
\left\lfloor\; x_{i,n+1} = x_{i,n} + \rho_na_{i,n}^* \right. \\
\text{for every } j \in J \\
\left\lfloor
\begin{array}{l}
y_{j,n+1} = y_{j,n} + \rho_nb_{j,n}^* \\
v_{j,n+1}^* = v_{j,n}^* + \rho_nc_{j,n}^*.
\end{array}
\right.
\end{array}
\right.
$$

Further connections

- Primal-dual splitting.
  - Consider the inclusion $0 \in Ax + L^*(B(Lx))$ and the associated Kuhn–Tucker operator

$$M\colon \mathcal{X} \times \mathcal{Y} \to 2^{\mathcal{X}\times\mathcal{Y}}\colon (x, y^*) \mapsto (Ax + L^*y^*) \times (-Lx + B^{-1}y^*).$$

  - The cutting plane method of (Alotaibi/PLC/Shahzad, 2014) and (PLC/Eckstein, 2018) generate points $(a_n, a_n^*) \in \operatorname{gra} A$ and $(b_n, b_n^*) \in \operatorname{gra} B$. This implicitly provides

$$(y_n, y_n^*) = \big((a_n, b_n^*), (a_n^* + L^*b_n^*, -La_n + b_n)\big) \in \operatorname{gra} M$$

    to construct $H_n \supset \operatorname{zer} M$.
  - The primal-dual framework of (Alotaibi/PLC/Shahzad, 2014) is therefore an instance of the Theorem with

$$K_n\colon (x, y^*) \mapsto \big(\gamma_n^{-1}x - L^*y^*, Lx + \mu_n y^*\big).$$

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- More generally, the block-iterative projective splitting method of (PLC/Eckstein, 2018) in an instance of the Theorem (Bùi, 2021).

PART 4:

Warped proximal iterations with Bregman kernels

Bregman forward-backward splitting

- $\mathcal{X}$ a reflexive real Banach space, $A\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ and $B\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ maximally monotone, and $f \in \Gamma_0(\mathcal{X})$ essentially smooth.
- $C = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{dom} A \subset \operatorname{int}\operatorname{dom} B$ and B is single-valued on $\operatorname{int}\operatorname{dom} B$.
- $(\forall x \in C)(\forall y \in C)(\forall z \in \mathscr{S})(\forall y^* \in Ay)(\forall z^* \in Az)$

$$\big\langle y - x, By - Bz \big\rangle \leqslant \kappa D_f(x,y) + \big\langle y - z, \delta_1(y^* - z^*) + \delta_2\big(By - Bz\big)\big\rangle.$$

- The objective is to

$$\text{find } x \in \mathscr{S} = (\operatorname{int}\operatorname{dom} f) \cap \operatorname{zer}(A+B) \neq \varnothing.$$

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- Apply the warped proximal point algorithm

$$x_{n+1} = J_M^{K_n} x_n$$

to $M = A + B$ with kernel $K_n = \gamma_n^{-1}\nabla f_n - B$ for a suitable essentially smooth function f_n.

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- The objective is to

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$$x_{n+1} = J_M^{K_n} x_n$$

to $M = A + B$ with kernel $K_n = \gamma_n^{-1}\nabla f_n - B$ for a suitable essentially smooth function f_n.
- We obtain the Bregman forward-backward splitting algorithm

$$x_{n+1} = \big(\nabla f_n + \gamma_n A\big)^{-1}\big(\nabla f_n(x_n) - \gamma_n B x_n\big).$$

Convergence

Theorem

"Under suitable assumptions,"

$$x_{n+1} = \big(\nabla f_n + \gamma_n A\big)^{-1}\big(\nabla f_n(x_n) - \gamma_n B x_n\big) \rightharpoonup x \in \mathscr{S}.$$

- This result provides, for instance, the convergence of the basic Bregman forward-backward splitting method

$$\big(\nabla f + \gamma A\big)^{-1}\big(\nabla f(x_n) - \gamma B x_n\big),$$

  which is new even in Euclidean spaces.
- It also allows us to recover and extend 4, so far unrelated, splitting frameworks.

$x_{n+1} = (\nabla f_n + \gamma_n A)^{-1}(\nabla f_n(x_n) - \gamma_n B x_n)$: Instantiations

- The iteration $x_{n+1} = (\nabla f + \gamma_n A)^{-1}(\nabla f(x_n))$ for finding a zero of A in a reflexive Banach space (Bauschke/Borwein/PLC, 2003).
- The iteration $x_{n+1} = (U_n + \gamma_n A)^{-1}(U_n x_n - \gamma_n B x_n)$ for finding a zero of $A + B$ in a Hilbert space, where U_n is a strongly positive Hermitian bounded linear operator (PLC/Vũ, 2014).
- The iteration

  $$x_{n+1} = (\nabla f + \gamma A)^{-1}(\nabla f(x_n) - \gamma B x_n)$$

  for finding a zero of $A + B$ in a Hilbert space, where f is real-valued and strongly convex (Renaud/Cohen, 1997).
- The iteration

  $$x_{n+1} = (\nabla f_n + \gamma_n \partial\varphi)^{-1}(\nabla f_n(x_n) - \gamma_n \nabla\psi(x_n))$$

  for minimizing $\varphi + \psi$ in a reflexive Banach space (Nguyen, 2017; see also Bauschke/Bolte/Teboulle, 2017).

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Bregman distance

- $f \in \Gamma_0(\mathcal{X})$ is a Legendre function if it is both (Bauschke/Borwein/PLC, 2001):
  - Essentially smooth: ∂f is both locally bounded and single-valued on its domain.
  - Essentially strictly convex: ∂f^* is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$.
- Take $f \in \Gamma_0(\mathcal{X})$, Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f \neq \varnothing$. The associated Bregman distance is

$$D_f\colon \mathcal{X} \times \mathcal{X} \to [0, +\infty]$$
$$(x, y) \mapsto \begin{cases} f(x) - f(y) - \langle x - y, \nabla f(y) \rangle, & \text{if } y \in \operatorname{int} \operatorname{dom} f; \\ +\infty, & \text{otherwise.} \end{cases}$$