Error Bounds for the Method of Simultaneous Projections with Infinitely Many Subspaces

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Introduction

$\mathcal{H}$ – real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

$r \in \{2, 3, \ldots\} \cup \{\infty\}$.

M_i – closed and linear subspaces of $\mathcal{H}$, $i = 1, \ldots, r$.

$M := \bigcap_{i=1}^r M_i$.

P_{M_i}, P_M – orthogonal projections onto M_i and M, respectively.

Approximate P_M by using $P_{M_1}, \ldots, P_{M_r}$.

Projection Methods

$$x_0 := x \in \mathcal{H}, \qquad x_k := T^k(x), \qquad k = 1, 2, \ldots,$$

$$\lim_{k \to \infty} \|x_k - P_{\operatorname{Fix} T}(x_0)\| = 0,$$

$$\operatorname{Fix} T \approx M.$$

Alternating/Cyclic Projection Method

$$T(x) := P_{M_r} \dots P_{M_1}(x), \quad \text{Fix}\, T = M.$$

Simultaneous Projection Method

$$T(x) := \sum_{i=1}^{r} \omega_i P_{M_i}(x), \quad \text{where} \quad \omega_i > 0, \quad \sum_{i=1}^{r} \omega_i = 1, \quad \text{Fix}\, T = M.$$

Douglas-Rachford Projection Method ($r = 2$)

$$T(x) := \frac{I + R_{M_2} R_{M_1}}{2}(x), \quad \text{where} \quad R_{M_i} := 2P_{M_i} - I, \quad i = 1, 2,$$

$$\text{Fix}\, T = M_1 \cap M_2 \oplus (M_1^{\perp} \cap M_2^{\perp}),$$

$$P_{M_2} P_{\text{Fix}\, T} = P_{M_1} P_{\text{Fix}\, T} = P_M$$

...

Outline

- Known error bounds when $r < \infty$.
- New error bounds for the simultaneous projection method when $r = \infty$.

Keywords

linear convergence, arbitrarily slow convergence, super-polynomially fast convergence, polynomial convergence

Known Error Bounds

$r < \infty$

Linear and Arbitrarily Slow Convergence

$$T = P_{M_r} \dots P_{M_1} \qquad T = \frac{1}{r}\sum_{i=1}^r P_{M_i} \qquad T = \frac{I + R_{M_2} R_{M_1}}{2}$$

Theorem 1. Exactly one of the following two statements holds:

(i) $\sum_{i=1}^r M_i^\perp$ is closed. Then T^k converges linearly to $P_{\operatorname{Fix} T}$.

(ii) $\sum_{i=1}^r M_i^\perp$ is not closed. Then T^k converges arbitrarily slowly to $P_{\operatorname{Fix} T}$.

CPM and **SPM** - Bauschke, Deutsch and Hundal, 2009 and 2010, Badea, Grivaux and Müller 2011; **DRPM** - Badea and Seifert 2017.

- $\overline{\sum_{i=1}^r M_i^\perp} = M^\perp$.
- T^k converges `linearly` to $P_{\operatorname{Fix} T}$ if there are $c > 0$ and $q \in (0,1)$ s.t.

$$\|T^k(x) - P_{\operatorname{Fix} T}(x)\| \leq cq^k \|x\|$$

  holds for each $k = 0, 1, 2, \dots$ and $x \in \mathcal{H}$.
- T^k converges `arbitrarily slowly` to $P_{\operatorname{Fix} T}$ if $T^k \to P_{\operatorname{Fix} T}$ and for each sequence $(a_k)_{k=0}^\infty \subset (0, \infty)$ satisfying $a_k \to 0$ as $k \to \infty$, there is x s.t.

$$\|T^k(x) - P_{\operatorname{Fix} T}(x)\| \geq a_k.$$

Linear and Arbitrarily Slow Convergence

$$T = P_{M_r} \dots P_{M_1} \qquad T = \frac{1}{r}\sum_{i=1}^r P_{M_i} \qquad T = \frac{I + R_{M_2} R_{M_1}}{2}$$

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- What is the optimal error bound in (i)?

$$\sup_{x \neq 0} \frac{\|T^k(x) - P_{\operatorname{Fix} T}(x)\|}{\|x\|} = \|T^k - P_{\operatorname{Fix} T}\|$$

- Are there any "good" starting points in (ii)? What are the error bounds?

Error Bounds when $r = 2$

$$M_1^\perp + M_2^\perp \text{ is closed} \quad \Longleftrightarrow \quad \cos(M_1, M_2) < 1$$

$$\cos(M_1, M_2) := \sup\{|\langle x_1, x_2\rangle| \colon x_i \in M_i \cap (M_1 \cap M_2)^\perp, \|x_i\| \leq 1, i = 1, 2\}$$

Theorem 2. $\|(P_{M_2} P_{M_1})^k - P_M\| = \cos(M_1, M_2)^{2k-1}$.

Aronszajn 1950 (inequality), Kayalar and Weinert 1988 (equality)

Theorem 3. $\left\| \left(\frac{P_{M_1} + P_{M_2}}{2} \right)^k - P_M \right\| = \left(\frac{1}{2} + \frac{1}{2} \cos(M_1, M_2) \right)^k$.

Reich, Z. 2017

Theorem 4. $\left\| \Big(\underbrace{\frac{I + R_{M_2} R_{M_1}}{2}}_{T} \Big)^k - P_{\operatorname{Fix} T} \right\| = \cos(M_1, M_2)^k$.

Bauschke, Bello Cruz, Nghia, Phan, Wang 2014

Error Bounds when $2 < r < \infty$

$$\sum_{i=1}^{r} M_i^{\perp} \text{ is closed} \quad \Longleftrightarrow \quad \cos(\boldsymbol{C}, \boldsymbol{D}) < 1$$

$$\boldsymbol{C} := M_1 \times \ldots \times M_r \quad \text{and} \quad \boldsymbol{D} := \{\underbrace{(x, \ldots, x)}_{r \text{ times}} : x \in \mathcal{H}\}$$

$$\boldsymbol{\mathcal{H}} := \bigoplus_{i=1}^{r} \mathcal{H}, \quad \langle \mathrm{x}, \mathrm{y} \rangle = \sum_{i=1}^{r} \langle x_i, y_i \rangle, \quad \|\mathrm{x}\| := \sqrt{\sum_{i=1}^{r} \|x_i\|^2}$$

Theorem 5. $\|(P_{M_r} \ldots P_{M_1})^k - P_M\| \leq \left(1 - \frac{1}{m^2}(1 - \cos(\boldsymbol{C}, \boldsymbol{D}))^2\right)^{k/2}$.

Badea, Grivaux, Müller 2011

Theorem 6. $\|(\frac{1}{r} \sum_{i=1}^{r} P_{M_i})^k - P_M\| = \cos(\boldsymbol{C}, \boldsymbol{D})^{2k}$.

Reich, Z. 2017

Super-Polynomially Fast Convergence

$$T = P_{M_r} \dots P_{M_1} \qquad T = \frac{1}{r}\sum_{i=1}^{r} P_{M_i} \qquad T = \frac{I + R_{M_2} R_{M_1}}{2}$$

Theorem 7. If $\sum_{i=1}^{r} M_i^{\perp}$ is not closed, then T^k converges super-polynomially fast to $P_{\text{Fix}\, T}$ on some dense linear subspace $X \subseteq \mathcal{H}$.

CPM and **DRPM** - Badea and Seifert 2016, 2017; **SPM** - Reich and Z. 2017.

- T^k converges `super-polynomially fast` to $P_{\text{Fix}\, T}$ on X if

$$k^n \| T^k(x) - P_{\text{Fix}\, T}(x) \| \to 0 \text{ as } k \to \infty$$

holds for all $x \in X$ and for all $n = 1, 2, \dots$.

Polynomial Convergence

Theorem 8. ($r \geq 2$) Assume $M = \{0\}$. For any $x \in \sum_{i=1}^{r} M_i^\perp$ there is $C(x) > 0$ s.t.

$$\|(P_{M_r} \dots P_{M_1})^k(x)\| \leq C(x) \cdot k^{-1/(4r\sqrt{r}+2)}, \quad k = 1, 2, \dots.$$

Theorem 9. ($r = 2$) Assume $M_1 \cap M_2 = \{0\}$. For any $x \in M_1^\perp + M_2^\perp$ there is $C(x) > 0$ s.t.

$$\|(P_{M_2} P_{M_1})^k(x)\| \leq \frac{C(x)}{\sqrt{k}}, \quad k = 1, 2, \dots.$$

Moreover, $\sqrt{k}$ cannot be replaced by $n^{1/2+\varepsilon}$ for any $\varepsilon > 0$.

Borodin and Kopecká 2020

New Error Bounds for the Simultaneous PM

$r = \infty$

S. Reich, R. Zalas, Error bounds for the method of simultaneous projections with infinitely many subspaces, *J. Approx. Theory* **272** (2021), 105648.

Simultanoeus Projection in $\mathcal{H}$

$$T_\omega(x) := \sum_{i=1}^{\infty} \omega_i P_{M_i}(x)$$

$$\omega := \{\omega_i\}_{i=1}^{\infty}, \quad \omega_i > 0,\ i = 1, 2, \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} \omega_i = 1$$

$$\operatorname{Fix} T_\omega = M$$

Product Space Setup

$$\boldsymbol{\mathcal{H}}_\omega := \left\{ \boldsymbol{x} = \{x_i\}_{i=1}^{\infty} \colon x_i \in \mathcal{H}, i = 1, 2, \ldots \text{ and } \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \right\}$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_\omega := \sum_{i=1}^{\infty} \omega_i \langle x_i, y_i \rangle \quad \text{and} \quad \|\boldsymbol{x}\|_\omega = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle_\omega}$$

Fact 10. The elements of $\boldsymbol{\mathcal{H}}_\omega$ can change if we change ω.

Alternating Projection in $\mathcal{H}_\omega$

$$\boldsymbol{C}_\omega := \left\{\{x_i\}_{i=1}^\infty : x_i \in M_i, i = 1, 2, \ldots \text{ and } \sum_{i=1}^\infty \omega_i \|x_i\|^2 < \infty\right\}$$

$$\boldsymbol{D}_\omega := \{\{x\}_{i=1}^\infty : x \in \mathcal{H}\}$$

Theorem 11. For each $\boldsymbol{x} = \{x\}_{i=1}^\infty \in \boldsymbol{D}_\omega$, we have

$$(P_{\boldsymbol{D}_\omega} P_{\boldsymbol{C}_\omega})^k(\boldsymbol{x}) = \{T_\omega^k(x)\}_{i=1}^\infty$$

and

$$P_{\boldsymbol{C}_\omega \cap \boldsymbol{D}_\omega}(\boldsymbol{x}) = \{P_M(x)\}_{i=1}^\infty.$$

Corollary 12. For each $x \in \mathcal{H}$ and $\boldsymbol{x} = \{x\}_{i=1}^\infty$, we have

$$\|T_\omega^k(x) - P_M(x)\| = \|(P_{\boldsymbol{D}_\omega} P_{\boldsymbol{C}_\omega})^k(\boldsymbol{x}) - P_{\boldsymbol{C}_\omega \cap \boldsymbol{D}_\omega}(\boldsymbol{x})\|_\omega \to 0.$$

Towards the Dichotomy Theorem

How to define "$\sum_{i=1}^{\infty} M_i^{\perp}$"?

$$\sum_{i=1}^{\infty} \omega_i M_i^{\perp} := \left\{ \sum_{i=1}^{\infty} \omega_i x_i : x_i \in M_i^{\perp} \text{ and } \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \right\}$$

$$\sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \quad \Longrightarrow \quad \sum_{i=1}^{\infty} \omega_i \|x_i\| < \infty \quad \Longrightarrow \quad \sum_{i=1}^{\infty} \omega_i x_i \in \mathcal{H}$$

The elements of $\sum_{i=1}^{\infty} \omega_i M_i^{\perp}$ can change if we change ω.

$$\overline{\sum_{i=1}^{\infty} \omega_i M_i^{\perp}} = M^{\perp}.$$

$$\sum_{i=1}^{\infty} \omega_i M_i^{\perp} \text{ is closed} \iff \cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega) < 1$$

$$\cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega) = \sup\left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M^{\perp},\ \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 \leq 1 \right\}$$

$$\cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega)^2 \leq \sup\left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M^{\perp},\ \|x_i\| \leq 1 \right\} \leq \cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega)$$

Theorem 13. The following conditions are equivalent:

(i) $\cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega) = 1$ for some sequence of weights $\omega = \{\omega_i\}_{i=1}^{\infty}$.

(ii) $\cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega) = 1$ for all sequences of weights $\omega = \{\omega_i\}_{i=1}^{\infty}$.

$$\sum_{i=1}^{\infty} \omega_i M_i^{\perp} \text{ is closed} \quad \Longleftrightarrow \quad \cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega) < 1$$

$$\cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega) = \sup\left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M^{\perp},\ \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 \leq 1 \right\}$$

$$\cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega)^2 \leq \sup\left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M^{\perp},\ \|x_i\| \leq 1 \right\} \leq \cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega)$$

Theorem 13. The following conditions are equivalent:

(i) $\sum_{i=1}^{\infty} \omega_i M_i^{\perp}$ is closed for some sequence of weights $\omega = \{\omega_i\}_{i=1}^{\infty}$.

(ii) $\sum_{i=1}^{\infty} \omega_i M_i^{\perp}$ is closed for all sequences of weights $\omega = \{\omega_i\}_{i=1}^{\infty}$.

Dichotomy

Theorem 14. Exactly one of the following two statements holds:

(i) $\sum_{i=1}^{\infty} \omega_i M_i^{\perp}$ is closed for all sequences of weights $\omega = \{\omega_i\}_{i=1}^{\infty}$. Then T_ω^k converges linearly to P_M.

(ii) $\sum_{i=1}^{\infty} \omega_i M_i^{\perp}$ is not closed for all sequences of weights $\omega = \{\omega_i\}_{i=1}^{\infty}$. Then T_ω^k converges arbitrarily slowly to P_M.

Optimal error bound in (i)? / "Good" starting points in (ii)?

Optimal error bound

Theorem 15. $\|T_\omega^k - P_M\| = \cos_\omega(\boldsymbol{C}_\omega, \boldsymbol{D}_\omega)^{2k}$.

Super-Polynomially Fast Convergence

Theorem 16. If $\sum_{i=1}^\infty \omega_i M_i^\perp$ is not closed, then T_ω^k converges super-polynomially fast to P_M on some dense linear subspace $X \subseteq \mathcal{H}$.

Polynomial Convergence

Theorem 17. For any $x \in \sum_{i=1}^\infty \omega_i M_i^\perp$ there is $C(x) > 0$ s.t.

$$\|T_\omega^k(x) - P_M(x)\| \leq \frac{C(x)}{\sqrt{k}}, \quad k = 1, 2, \ldots.$$

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Thank you for your attention!