Generalized Bregman Distances

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Workshop on Optimization and Operator Theory

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Dedicated to Professor Lev Bregman

University of South Australia

Outline

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This talk contains joint work with:

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Federation University, Ballarat, Australia

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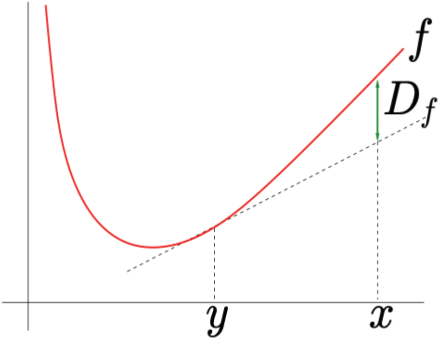
Hong Kong Polytechnic and Curtin University, Perth, Australia

$f : X \to \mathbb{R}_{+\infty}$ strictly convex, smooth, (Bregman, 1967)

Bregman distance : $D_f(x,y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle$



Using specific choices of f, we obtain:

Kullback-Liebler divergence A distance between positive vectors, used in information theory, statistics, portfolio selection,...

Itakura-Saito divergence A non-symmetric measure of difference between probability distributions. Used to measure sound quality and speech processing

Squared Euclidean (ℓ_2) distance for $f(\cdot) := (1/2)\|\cdot\|^2$ we have $D_f(x, y) = (1/2)\|x - y\|^2$

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What is the Bregman distance really measuring?

$$D_f(x,y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle =$$

$$(Fenchel - Young) = f(x) + f^*(\nabla f(y)) - \langle \nabla f(y), x \rangle$$

Hence, $\boxed{D_f(x,y) = 0 \iff \nabla f(x) = \nabla f(y)}$

so D_f measures distance between images of the gradient map!

If f is smooth, then

$$D_f(x,y) \quad \geq \quad 0 \;\; \forall x,y \iff f \text{ is convex}$$

Due to a "Fenchel-Young" property for the graph of ∇f:

$$f(x) + f^*(v) \geq \langle x, v\rangle \qquad \forall x \in X,\; v \in X^*$$

$$f(x) + f^*(v) = \langle x, v\rangle \iff v = \nabla f(x)$$

An analogous property holds for max-mon T!

The family $\mathcal{H}(T)$ Introduced in [RSB-Svaiter, 2002]

For $T : X \rightrightarrows X^*$ max-mon, the Fitzpatrick family $\mathcal{H}(T)$ consists of functions $h : X \times X^* \to \mathbb{R}_{+\infty}$ convex and norm-weak* lsc, s.t.

$$h(x, v) \geq \langle x, v\rangle \quad \forall x \in X,\ v \in X^*$$
$$h(x, v) = \langle x, v\rangle \iff v \in Tx,$$

Fenchel-Young gives $h(x, v) := f(x) + f^*(v) \in \mathcal{H}(\partial f)$

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$\mathcal{H}(T)$ and the Fitzpatrick function

$\mathcal{H}(T)$ has a smallest and a biggest element. The smallest one is the *Fitzpatrick function*[1]:

$$\mathcal{F}_T(x,v) := \sup_{(z,w)\in G(T)} \langle z-x, v-w\rangle + \langle x, v\rangle.$$

The biggest is $\sigma_T = cl\,conv(\pi + \delta_{G(T)})$ and in fact

$$\sigma_T(x,v) := \mathcal{F}_T{}^*(v,x),$$

where $\pi := \langle\cdot,\cdot\rangle$.

[1] Fitzpatrick, 1988.

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Extending the Bregman distance to a generic T

For T max-mon, $x, y \in D(T)$, and $h \in \mathcal{H}(T)$, recall that

$$\begin{aligned} h(x,v) &\geq \langle x, v\rangle \quad \forall x \in X,\ v \in X^* \\ h(x,v) &= \langle x, v\rangle \iff v \in Tx, \end{aligned}$$

The "sharp" version $D_T^{\sharp,h}(x,y) := \sup_{v\in Ty} h(x,v) - \langle x, v\rangle$

The "flat" version $D_T^{\flat,h}(x,y) := \inf_{v\in Ty} h(x,v) - \langle x, v\rangle$

Note that the distance depends on the choice of h!

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Particular case 1: $T = \nabla f$ and $h = h^{FY} = f + f^*$

If f convex and smooth, choose then

$$D^{\sharp,h^{FY}}_{\nabla f}(x,y) = D^{\flat,h^{FY}}_{\nabla f}(x,y) = D_f(x,y),$$

for $(x,y) \notin (\operatorname{dom} f \setminus \operatorname{dom} \nabla f) \times \operatorname{dom} \nabla f$ (region where coincides with the classical Bregman distance)

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Particular case 2: T max-mon and $h = \mathcal{F}_T$

The "sharp" version $D_T^{\sharp,h}(x,y) := \sup_{v \in Ty} \mathcal{F}_T(x,v) - \langle x, v \rangle$

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where $\mathcal{F}_T(x,v) := \sup_{(z,w) \in G(T)} \langle z - x, v - w \rangle + \langle x, v \rangle.$

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$T = \nabla f$ and $h = \mathcal{F}_{\nabla f}$

Example 1. Negative Burg entropy

$f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(t) = \begin{cases} -\log t, & t > 0 \\ +\infty, & \text{c.c.} \end{cases}$$

$$D_{\nabla f}^{\mathcal{F}_{\nabla f}}(x, y) = \begin{cases} \left(\sqrt{\frac{x}{y}} - 1\right)^2, & x \geq 0,\ y > 0 \\ +\infty, & \text{c.c.} \end{cases}$$

Used $\mathcal{F}_{\nabla f}$ from [Bauschke-McLaren-Sendov, 2005]

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Example 2: negative Boltzmann-Gibbs-Shannon/Kullback-Leibler/ entropy

$$f : \mathbb{R} \to \mathbb{R} \text{ defined as } f(t) = \begin{cases} t\log t - t, & t > 0 \\ 0 & t = 0 \\ +\infty, & \text{c.c.} \end{cases}$$

$$D_{\nabla f}^{\mathcal{F}_{\nabla f}}(x,y) = \begin{cases} \frac{y}{e}, & x = 0,\ y \geq 0 \\ x\left[W(e\,\frac{x}{y}) + \frac{1}{W(e\,\frac{x}{y})} - 2\right], & x > 0,\ y > 0 \\ +\infty, & \text{c.c.} \end{cases}$$

$W : \mathbb{R}_{++} \to \mathbb{R}_{++}$ inverse of $t \to te^t$ (Lambert-W function). Used $\mathcal{F}_{\nabla f}$ from [Bauschke-McLaren-Sendov, 2005]

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Prox-envelopes: Moreau, Bregman, Fitzpatrick

The Moreau prox-envelope of a convex $\theta : \mathbb{R} \to \mathbb{R}$:

$$e^M_\gamma(\theta)(x) := \inf_w \left\{ \theta(w) + \frac{1}{\gamma}\|w - x\|^2 \right\}$$

The Bregman prox-envelope:

$$e^B_\gamma(\theta)(x) := \inf_w \left\{ \theta(w) + \frac{1}{\gamma} D_f(w, x) \right\},$$

where $D_f(x, y) = D^{h^{FY}_f}_{\nabla f}(x, y) = D^{(f+f^*)}_{\nabla f}(x, y)$ classical Bregman distance

A new family of Prox-envelopes

The Fitzpatrick prox-envelope:

$$e_\gamma^F(\theta)(x) := \inf_w \left\{ \theta(w) + \frac{1}{\gamma} D_{\nabla f}^{\mathcal{F}_{\nabla f}}(w,x) \right\}$$

or, in general, taking $D_T^{\star,h} \in \{D_T^{\sharp,h}, D_T^{\flat,h}\}$:

$$e_\gamma^{\star,h,T}(\theta)(x) := \inf_w \left\{ \theta(w) + \frac{1}{\gamma} D_T^{\star,h}(w,x) \right\}$$

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Asymptotic behaviour when $\gamma \downarrow 0$:

- θ convex, proper, lsc, $y \in \operatorname{dom} T \cap \operatorname{dom} \theta$
- $(\operatorname{dom} T \cap \operatorname{dom} \theta) \times \{y\} \subseteq \operatorname{dom} \mathcal{D}_T^{\sharp,h}$
- $\varphi_\mu(\cdot) := \theta(\cdot) + \frac{1}{\mu}\mathcal{D}_T^{\sharp,h}(\cdot, y)$ is coercive for some $\mu \in \mathbb{R}_{++}$.

Let $s_\gamma \in \operatorname{Argmin} \varphi_\gamma$. Then, as $\gamma \downarrow 0$,

- $\mathcal{D}_T^{\sharp,h}(s_\gamma, y) \to 0$.
- If T is str. mon. over $\operatorname{dom} T \cap \operatorname{dom} \theta$,

$$s_\gamma \to y, \quad e_\gamma^{\sharp,h,T}(\theta)(y) \uparrow \theta(y), \quad \theta(s_\gamma) \to \theta(y), \quad \frac{1}{\gamma}\mathcal{D}_T^{\sharp,h}(s_\gamma, y) \to 0.$$

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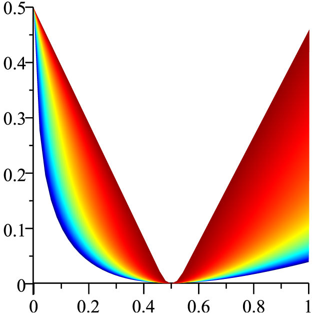
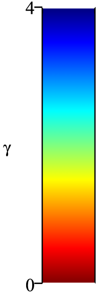
Bregman vs. Fitzpatrick envelope for $\theta(t) := |t - (1/2)|$ for Kullback-Leibler f, $T = \log(\cdot)$



Bregman envelope
$h = f + f^*$



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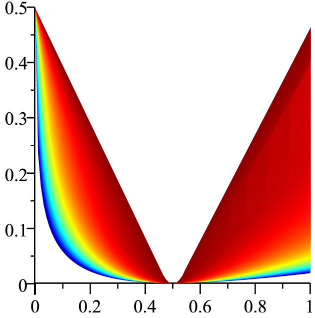

Fitzpatrick envelope
$h = \mathcal{F}_{\nabla f}$

*Credit for the figures: Scott Lindstrom

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Bregman distance between two set-valued maps

$T, S : X \rightrightarrows X^*$, T max-mon, S gral., $h_T \in \mathcal{H}(T)$:

The "sharp" version

$$D_S^{\sharp, h_T}(x, y) := \sup_{v \in Sy} h_T(x, v) - \langle x, v \rangle$$

The "flat" version

$$D_S^{\flat, h_T}(x, y) := \inf_{v \in Sy} h_T(x, v) - \langle x, v \rangle$$

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A new measure of degree of overlap between Tx and Sy

T max-mon, $h \in \mathcal{H}(T)$, $(x, y) \in D(T) \times D(S)$:

(a) If S loc. bded in $\mathrm{int} D(S)$, weakly-closed valued,

$(x, y) \notin \mathrm{bdry} D(S) \times \mathrm{bdry} D(T)$.

$$D_S^{\flat,h}(x, y) = 0 \iff Sy \cap Tx \neq \emptyset$$

(b) $D_S^{\sharp,h}(x, y) = 0 \iff Sy \subset Tx$

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T max-mon, $h \in \mathcal{H}(T)$, $u \in X^*$: Find x s.t. $u \in Tx$ (P_T)

If (P_T) difficult, find y sol. of the "better conditioned" problem

$$(P_S) \qquad u \in Sy$$

For this y, find x s.t. $D_S^{\sharp,h}(x,y) \leq \varepsilon$, then x solves

$$(P_{T,\varepsilon}) \qquad u \in T^e(\varepsilon, x),$$

where

$$T^e(\varepsilon, x) := \{v \in X^* \ : \ \langle x - y, v - w\rangle \geq -\varepsilon \ \forall (y,w) \in G(T)\}$$

Note: $D_S^{\sharp,h}(\cdot, y)$ is convex and x solves an ε-approximation of (P_T)!

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An enlargement of T induced by $h \in \mathcal{H}(T)$

T max-mon, the *enlargement* L^h of T is

$$L^h(\varepsilon, x) := \{v \in X^* : h(x, v) - (x, v) \leq \varepsilon\}.$$

Consider the problem:

(PS) find $x \in X$ s.t. $0 \in Sx + Tx$.

GBDs define approximate solutions of (PS) using L^h

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T max-mon, S point-to-set, $h \in \mathcal{H}(T)$

Consider the following statements:

(a) $0 \in L^h(\varepsilon, x) + Sx$.

(b) $\mathcal{D}^{\flat,h}_{-S}(x,x) \leq \varepsilon$.

Then (a) $\Longrightarrow$ (b) (necessary condition for optimality). Moreover, if $\operatorname{dom} S$ is open and S is locally bounded with weakly closed images, then the two statements are equivalent.

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Some Questions

(a) How can we use these new distances for obtaining more efficient solution techniques for variational inequalities/inclusion problems

(b) When $T = S = \partial f$, will the Fitzpatrick distances play a role similar to that of the classical Bregman distances (Bregman projections, convergence analysis, etc)?

(c) Can these distances be used to regularize/penalize in prox-like iterations for variational inequalities?

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Conclusions

Convex functions appear naturally when studying maximally monotone operators.

Some notions involving convex functions (such as classical Bregman distances), can be extended to maximally monotone operators, thus producing new tools both for convex analysis and for maximal monotone theory.

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