Nonlinear Production-Consumption Equilibrium (Theory æ Methods)

Roman A.Polyak
rpolyak@gmu.edu

Depatment of Mathematics
Technion, Israel

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Linear Programming

Kantorovich L.V. 1939, George Dantzig 1947:

$$(c, x^*) = \max\{(c, x) | Ax \leq b,\ x \geq 0\}$$
$$A : \mathbb{R}^n \to \mathbb{R}^m, b \in \mathbb{R}^m, c \in \mathbb{R}^n$$

John von Neumann 1947:

$$(b,\lambda^*) = \min\{(b, \lambda) | A^T\lambda \geq c, \lambda \geq 0\}$$

1. $(c, x^*) = (b, \lambda^*)$
2. $(A^T\lambda^* - c, x^*) = 0, (b - Ax^*, \lambda^*) = 0$

Linear Equilibrium

In 1975 L.V. Kantorovich and T.C. Koopmans
Shared the Nobel Prize in Economics
"for their contribution to the theory of optimal allocation of limited resources."

G.Dantzig (1914-2005)
L.Kantorovich (1912-1986)
T.Koopmans (1910-1985)

Input-Output Model

$$A : \mathbb{R}^n \to \mathbb{R}^n, \quad c = (I - A)x$$

For a given $c \in \mathbb{R}^n_+$ find x_c:

$$x_c = (I - A)^{-1} c$$

$$q = (I - A)^T \lambda$$

For a given $q \in \mathbb{R}^n_+$ find λ_q :

$$\lambda_q = \left((I - A)^T \right)^{-1} q$$

For a productive economy

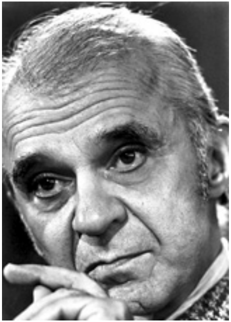
$$\sum_{i=1}^{n} a_{i,j} < 1 \quad , 1 \leq j \leq n$$

$$\Downarrow$$

$$x_c \in \mathbb{R}^n_{++}, \quad \lambda_q \in \mathbb{R}^n_{++}$$

Wassily Leontief

(1906-1999)



In 1973 Wassily Leontief received the Nobel Prize in Economics "for the development of the input-output(IO) model and for its application to the important economic problems" .

LP does not have a consumption part

IO does not have a factor part

Moreover

1. The main ingredients of LP: vectors $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are **fixed and given a priori**.
2. The same is true for the main ingredients of IO model: vectors $c \in \mathbb{R}^n_+$ and $q \in \mathbb{R}^n_+$ are **fixed and given a priori**.

Nonlinear Production-Consumption Equilibrium (NPCE).

1. NPCE takes into account both consumption and resources.
2. The fixed production cost vector p is replaced by production operator $p : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, that $x \to p(x)$.
3. The fixed consumption vector c is replaced by consumption operator $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, that $\lambda \to c(\lambda)$.
4. The fixed factors (resources) vector r is replaced by factors operator $r : \mathbb{R}^m_+ \to \mathbb{R}^m_+$, that $v \to r(v)$.
5. $A : \mathbb{R}^n \to \mathbb{R}^n, B : \mathbb{R}^n \to \mathbb{R}^m$,
   $X \in \mathbb{R}^n_+, \Lambda \in \mathbb{R}^n_+, V \in \mathbb{R}^m_+$.

NPCE

$y^* = (x^*, \lambda^*, v^*) \in \Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m_+$:

$$x^* \in \operatorname{Argmin}\Big\{ (p(x^*), X) \,|(I - A)X \geq c(\lambda^*), \quad BX \leq r(v^*), X \in \mathbb{R}^n_+ \Big\} \tag{1}$$

$$(\lambda^*, v^*) \in \operatorname{Argmax}\Big\{ (c(\lambda^*), \Lambda) - (r(v^*), V) \,\Big|\, (I - A)^T\Lambda - B^T V \leq p(x^*), \Lambda \in \mathbb{R}^n_+, V \in \mathbb{R}^m_+ \Big\} \tag{2}$$

Walras-Wald Equilibrium

$$\Omega = \left\{x \in \mathbb{R}^n_+ : \; Ax \leq b\right\}$$

$$x^* \in \Omega : (c(x^*), x^*) = \max\left\{(c(x^*), x) : x \in \Omega\right\}$$

If such $x^* \in \Omega$ exists, then there is a solution for the dual problem

$$(b, \lambda^*) = \min\left\{(b, \lambda) : A^T\lambda \geq c(x^*), \lambda \in \mathbb{R}^m_+\right\}$$

and

1. $(c(x^*), x^*) = (b, \lambda^*)$
2. $(b - Ax^*, \lambda^*) = 0$
3. $\left(A^T\lambda^* - c(x^*), x^*\right) = 0$

L.Walras (1874), A.Wald (1935-1936), H.Kuhn (1956),

S.Zuchovitsky ,R.Polyak, M.Primak 1969,1970,1973

A.Bakushinskij and B.Polyak (1974).

Assuming that NPCE exists, then:

1. The total production cost reaches its minimum $(p(x^*), x^*)$.
2. The total consumption without the cost of factors $(c(\lambda^*), \lambda^*) - (r(v^*), v^*)$ reaches its maximum.
3. $(p(x^*), x^*) + (r(v^*), v^*) = (c(\lambda^*), \lambda^*)$.

Lagrangian for the LP(1)

$$\mathcal{L}(y^*, X, \Lambda, V) = (p(x^*), X) - (\Lambda, (I - A)X - c(\lambda^*)) \\ - (V, -BX + r(v^*))$$

$$y^* \in \operatorname*{Argmin}_{X \in \mathbb{R}^n_+} \max_{\substack{\Lambda \in \mathbb{R}^n_+ \\ V \in \mathbb{R}^m_+}} \mathcal{L}(y^*; X, \Lambda, V)$$

$$\omega(y) = \operatorname*{Argmin}_{X \in \mathbb{R}^n_+} \max_{\substack{\Lambda \in \mathbb{R}^n_+ \\ V \in \mathbb{R}^m_+}} \mathcal{L}(y; X, \Lambda, V)$$

NPCE $y^* \in \Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m_+$:

$$y^* \in \omega(y^*)$$

Finding NPCE $\Leftrightarrow$ Solving VI

$$x^* \in \text{Argmin}\left\{\mathcal{L}(y^*, X, \lambda^*, v^*) | X \in \mathbb{R}^n_+\right\} =$$
$$= \text{Argmax}\left\{\left((I-A)^T\lambda^* - p(x^*) - B^T v^*, X\right) \middle| X \in \mathbb{R}^n_+\right\}$$

$$\Downarrow$$

$$(I-A)^T\lambda^* - p(x^*) - B^T v^* \leq 0. \tag{3}$$

$$(\lambda^*, v^*) \in \text{Argmax}\left\{\mathcal{L}(y^*, x^*, \Lambda, V) | \Lambda \in \mathbb{R}^n_+, V \in \mathbb{R}^m_+\right\} =$$
$$= \text{Argmax}\left\{ (c(\lambda^*) - (I-A)x^*, \Lambda) + (Bx^* - r(v^*), V) \middle| \Lambda \in \mathbb{R}^n_+, V \in \mathbb{R}^m_+ \right\}.$$

Therefore

$$c(\lambda^*) - (I-A)x^* \leq 0, \qquad Bx^* - r(v^*) \leq 0. \tag{4}$$

Consider operator $g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$:

$$g(y) = \Big((I - A)^T \lambda - p(x) - B^T \lambda; \quad c(\lambda) - (I - A)x; Bx - r(v)\Big) \tag{5}$$

$$g(y^*) \le 0, \tag{6}$$

therefore $y^* \in \Omega$ - solution of VI

$$(g(y^*), Y - y^*) \le 0, \quad \forall Y \in \Omega \tag{7}$$

because of (6) and complementarity condition $(g(y^*), y^*) = 0$

Theorem 1

Finding NPCE is equivalent to solving VI (7) with nonlinear operator g given by (5) and feasible set $\Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m_+$.

Operators and their properties

1. For any x_1 and $x_2 \in \mathbb{R}^n_+$ there is $\alpha > 0$, that
$(p(x_1) - p(x_2), x_1 - x_2) \geq \alpha \left\| x_1 - x_2 \right\|^2, \quad \|u\| = (u, u)^{\frac{1}{2}}$.

2. For any λ_1 and $\lambda_2 \in \mathbb{R}^n_+$ there is $\beta > 0$, that
$(c(\lambda_1) - c(\lambda_2), \lambda_1 - \lambda_2) \leq -\beta \left\| \lambda_1 - \lambda_2 \right\|^2$.

3. For any v_1 and $v_2 \in \mathbb{R}^m_+$ there is $\gamma > 0$, that
$(r(v_1) - r(v_2), v_1 - v_2) \geq \gamma \left\| v_1 - v_2 \right\|^2$.

$$\Downarrow$$

Lemma 1

For $\delta = \min\{\alpha, \beta, \gamma\} > 0$ and for any y_1 and y_2 from Ω we have

$$(g(y_1) - g(y_2), y_1 - y_2) \leq -\delta \left\| y_1 - y_2 \right\|^2 \tag{8}$$

Finding NPCE as a two person concave game

The payoff function for the first player

$$\varphi_1(y; X, \lambda, v) = -\mathcal{L}(y; X, \lambda, v)$$

with strategy $X \in \mathbb{R}^n_+$

The payoff function for the second player

$$\varphi_2(y; x, \Lambda, V) = \mathcal{L}(y; x, \Lambda, V)$$

with strategy $(\Lambda; V) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$

The normalized payoff function

$$\Phi(y, Y) = \varphi_1(y; X, \lambda, v) + \varphi_2(y; x, \Lambda, V)$$

Let's consider the following map

$$y \to \omega(y) = \text{Argmax}\left\{\Phi(y; Y)\middle| Y \in \Omega\right\}$$

Then $y^* \in \Omega$:

$$y^* \in \omega(y^*) \tag{9}$$

Pseudo-gradient

$$g(y) = \Delta_Y \Phi(y, Y)_{|Y=y} =$$
$$= \left\{(I - A)^T \lambda - p(x) - B^T v; c(\lambda) - (I - A)x; Bx - r(v)\right\}$$

$$\begin{array}{ccc} y^* \in \Omega \text{ - solution of VI(7)} & \Longleftrightarrow & y^* \text{ - fix point (9)} \\ \Updownarrow & & \Updownarrow \\ & y^* \text{ - solution of two person concave game.} & \end{array}$$

$\Phi(y, Y)$ is linear in Y and Ω is unbounded set, therefore existence y^* can't be proven using standard considerations and Kakutani's Theorem.

Theorem 2

If p, c and r continuous and (8) holds, then there exists $y^* \in \Omega$, that $y^* \in \omega(y^*)$ and y^* is unique.

We assume

$$\|g(y_1) - g(y_2)\| \leq L \|y_1 - y_2\| \tag{10}$$

for any $y_1, y_2 \in \Omega$.

Projection on Ω

$$u \in \mathbb{R}^q, \quad v = \mathrm{Pr}_{\mathbb{R}^q_+}(u) = [u]_+;$$

$$[u]_+ = ([u_1]_+, \ldots, [u_q]_+)$$

$$[u_j]_+ = \begin{cases} u_j & , \quad u_j \geq 0 \\ 0 & , \quad u_j < 0 \end{cases} , 1 \leq j \leq q$$

Projection Operator

$$P_\Omega(y) = ([x]_+, [\lambda]_+, [v]_+)$$

1. $\|P_\Omega(u) - P_\Omega(v)\| \leq \|u - v\|$
2. $y^* = P_\Omega\left(y^* + tg(y^*)\right) \ , \quad \forall t > 0$

y^* - fixed point of the map

$$P_\Omega(I + tg) \ , \quad \forall t > 0$$

Pseudo-gradient projection (PGP) method

$$y_0 \in \Omega, \quad y_s \Rightarrow y_{s+1}:$$

$$y_{s+1} = [y_s + tg(y_s)]_+ \ , \quad t > 0$$

$$x_{s+1,j} = \left[x_{s,j} + t\left((I - A)^T \lambda_s - p(x_s) - B^T v_s \right)_j \right]_+ \tag{11}$$

$$\lambda_{s+1,j} = \left[\lambda_{s,j} + t\left(c(\lambda_s) - (I - A)x_s \right)_j \right]_+ \tag{12}$$

$$v_{s+1,i} = [v_{s,i} + t\left(Bx_s - r(v_s) \right)_i]_+ \tag{13}$$

Euler's Method for

$$\frac{dx}{dt} = (I - A)^T \lambda - p(x) - B^T v$$

$$\frac{d\lambda}{dt} = c(\lambda) - (I - A)x$$

$$\frac{dv}{dt} = Bx - r(v)$$

Example

$$p(x) = \nabla \left(\frac{1}{2} x^T P x + q^T x \right) \quad P \succeq 0$$

$$c(\lambda) = \nabla \left(\frac{1}{2} \lambda^T C \lambda + d^T \lambda \right) \quad C \preceq 0$$

$$r(v) = \nabla \left(\frac{1}{2} v^T R v + s^T v \right) \quad R \succeq 0$$

$$(g(y) - g(y^*), y - y^*) \leq -\delta \left\| y - y^* \right\|^2 \tag{14}$$

$$\left\| g(y) - g(y^*) \right\| \leq L \left\| y - y^* \right\| \tag{15}$$

$$0 < \varkappa = \mathrm{cond}(g) = \delta L^{-1} < 1$$

Theorem 3

Under condition (14)-(15) the PGP method produces a sequence $\{y_s\}_{s=1}^{\infty}$*:*

1. *for any* $0 < t < 2\delta L^{-2}$ *the sequence* $\{y_s\}_{s=1}^{\infty}$ *converges to* y^* *and for* $0 < q(t) = (1 - 2t\delta + t^2L^2)^{1/2} < 1$ *we have*

$$\|y_{s+1} - y^*\| \leq q(t)\,\|y_s - y^*\|\,; \tag{16}$$

2. *for* $t = \delta L^{-2} = \min\{q(t)|t > 0\}$

$$\|y_{s+1} - y^*\| \leq (1 - \varkappa^2)^{1/2}\,\|y_s - y^*\|\,, \quad s \geq 1; \tag{17}$$

3.

$$comp(PGP) = O(n^2\varkappa^{-2}\ln\epsilon^{-1}) \tag{18}$$

operations, where $\epsilon > 0$ *is the required accuracy.*

$$\boldsymbol{\delta = 0 \text{ - ?}}$$

Extra pseudo-gradient (EPG) method

Prediction phase

$$\hat{y}_s = P_\Omega(y_s + tg(y_s)) = [y_s + tg(y_s)]_+ \tag{19}$$

Correction phase

$$y_{s+1} = P_\Omega\left(y_s + tg\left(\hat{y}_s\right)\right) = [y_s + tg(\hat{y}_s)]_+ \tag{20}$$

G. Korpelevich (1976)

Galya Korpelevich with her son Misha Polyak,
currently Math. Professor at the Technion

In the **prediction** phase one **predicts** production vector

$$\hat{x}_s = \left[x_s + t\left((I - A)^T \lambda_s - p(x_s) - B^T v_s\right)\right]_+ \tag{21}$$

consumption prize vector

$$\hat{\lambda}_s = [\lambda_s + t(c(\lambda_s) - (I - A)x_s)]_+ \tag{22}$$

and factor price vector

$$\hat{v}_s = [v_s + t(Bx_s - r(v_s))]_+ \tag{23}$$

In the **correction** phase one finds the new approximation y_{s+1}:

$$x_{s+1} = \left[x_s + t\left((I-A)^T\hat{\lambda}_s - p(\hat{x}_s) - B^T\hat{v}_s\right)\right]_+ \quad (24)$$

$$\lambda_{s+1} = \left[\lambda_s + t\left(c(\hat{\lambda}_s) - (I-A)\hat{x}_s\right)\right]_+ \quad (25)$$

$$v_{s+1} = [v_s + t\,(B\hat{x}_s - r(\hat{v}_s))]_+ \quad (26)$$

Theorem 4

If p, c and r monotone operators, which satisfy Lipschitz condition on Ω, then for any $0 < t < \left(\sqrt{2}L\right)^{-1}$ the EPG method generate a converging sequence $\{y_s\}_{s=1}^{\infty} : \lim\limits_{s\to\infty} y_s = y^*$

Local strong monotonicity

$$(p(x) - p(x^*), x - x^*) \geq \alpha \|x - x^*\|^2 , \tag{27}$$

$$(c(\lambda) - c(\lambda^*), \lambda - \lambda^*) \leq -\beta \|\lambda - \lambda^*\|^2 , \tag{28}$$

$$(r(v) - r(v^*), v - v^*) \geq \gamma \|v - v^*\| . \tag{29}$$

Local Lipschitz condition

$$\|g(y) - g(y^*)\| \leq L \|y - y^*\| , \quad \forall y \in \Omega. \tag{30}$$

Lemma 2

If (27)-(29) are satisfied, then there is $\delta > 0$, *that for any* $y \in \Omega$

$$(g(y) - g(y^*), y - y^*) \leq -\delta \|y - y^*\|^2 . \tag{31}$$

Theorem 5

If conditions (30) and (31) are satisfied, then

1. *for* $\nu(t) = 1 + 2\delta t - 2(Lt)^2$ *and* $0 < t < \left(\sqrt{2}L\right)^{-1}$ *we have* $0 < q(t) = 1 - 2\delta t + 4(\delta t)^2 \left(\nu(t)\right)^{-1} < 1$ *and the following bound*

   $$\|y_{s+1} - y^*\| \leq \sqrt{q(t)}\, \|y_s - y^*\|$$

   holds;
2. $q\left((2L)^{-1}\right) = (1 + \varkappa)(1 + 2\varkappa)^{-1}$;
3. *for any* $\varkappa \in (0, 0.5)$ *the following bound*

   $$\|y_{s+1} - y^*\| \leq \sqrt{1 - 0.5\varkappa}\, \|y_s - y^*\|$$

   holds;
4. *comp(EPG)* $\leq O(n^2 \varkappa^{-1} \ln \epsilon^{-1})$*,* *where* $\epsilon > 0$ *- required accuracy.*

Generalized Walras law

$$(c(\lambda^*), \lambda^*) = (p(x^*), x^*) + (r(v^*), v^*),$$

$$x_j^* > 0 \Rightarrow p_j(x^*) + \left(B^T v^*\right)_j - \left((I - A)^T \lambda^*\right)_j = 0$$
$$x_j^* = 0 \Leftarrow p_j(x^*) + \left(B^T v^*\right)_j - \left((I - A)^T \lambda^*\right)_j > 0,$$

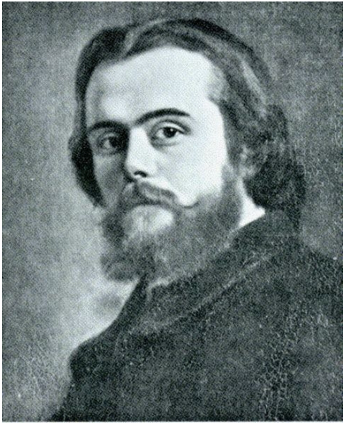
$$\lambda_j^* > 0 \Rightarrow ((I - A)x^*)_j - c_j(x^*) = 0$$
$$\lambda_j^* = 0 \Leftarrow ((I - A)x^*)_j - c_j(x^*) > 0,$$

$$v_i^* > 0 \Rightarrow r_i(v^*) - (Bx^*)_i = 0$$
$$v_i^* = 0 \Leftarrow r_i(x^*) - (Bx^*)_i > 0.$$

Market clearing

Marie-Esprit-Léon Walras

(1834-1910)



Leon Walras has been hailed by Dr. William Jaffe (1898-1980), the leading historian of economic thought, as one of the "greatest of all economists". Walras's most admired work "Elements of Pure Economics" was published in 1874.

Walras set forth the "Neoclassical Theory in a formal general equilibrium setting and is currently considered the Father of General Equilibrium Theory".

In the aftermath of the "Elements...", Walras built up a correspondence with virtually every important economist of the time from America to Russia, but for the most part he was largely ignored or dismissed by both economic and mathematical mainstream.

R.Polyak "Introduction to Continuous Optimization", Springer, 2021, 541 pages

R.Polyak "Nonlinear Equilibrium for Resource Allocation Problems", Contemporary Mathematics, AMS, 2015

Happy Birthday Lev