A Multidimensional Approach to Non-Cooperative Strategic Games

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Agenda

1. An Outlook on Equilibrium Problems of Non-Cooperative Strategic Games

The study of an equilibrium problem was started by Cournot (1838), involved with an oligopolistic economy. However, Nash (1950,1951) introduced this concept formally. Subsequently, Arrow and Debreu (1954) extended it to generalized Nash equilibrium problem which is useful in mathematical modelings of various economic world problems, for instance, routing problems in communication networks, some design problems arise in engineering applications, oligopolistic market problems, environmental models and many more.

$N-$Person Non-Cooperative Game: Nash Equilibrium Problem

- There are N players; each called player $w(w = 1, 2, \ldots, N)$,
- x^w: vector of player w's strategies,
- x^{-w}: vector of strategies of all players except player w,

- K_w: set of player w's strategies,
- $f^w(x^w, x^{-w})$: cost function/utility function/payoff function of the player w.
- Player w's optimization problem:

$$\underset{x^w}{\text{Min}}\, f^w(x^w, x^{-w})$$

$$\text{such that } x^w \in K_w.$$

- A tuple of N players strategies $y = (y^1, y^2, \ldots, y^{w-1}, y^w, y^{w+1}, \ldots, y^N)$ is called a **Nash equilibrium** if for each $w = 1, 2, \ldots, N$, y^w is an optimal solution of the above-mentioned optimization problem.

$N-$Person Non-Cooperative Game: Generalized Nash Equilibrium Problem

- In the classical Nash game, the strategy set of each player is independent of the other player's strategies.
- We may consider a more general situation where the strategy set of a player is dependent on the other player's strategies through some joint constraints.
- This may sound strange, since it is assumed that each player must know the other player's strategies to know his/her feasible strategy set. In other words, it is supposed that the players choose their strategies independently of each other, and then the constraints involving those strategies must somehow be satisfied.
- However one may imagine, for example, that there is a 'regulator' in the game who can impose taxes on players, thereby forcing their actions to satisfy the constraints.

- K: a nonempty closed and convex set, (just for an example)
  $K = \{x = (x^1, x^2, \ldots, x^{w-1}, x^w, x^{w+1}, \ldots, x^N) : \sum_{w=1}^{N} x^w < \rho\}$,
- $K_w(x^{-w})$: strategy set of the player w that is allowed to depend on the rival players' strategies vector x^{-w}, (just for an example)

$$K_w(x^{-w}) = \{x^w : (x^w, x^{-w}) \in K\}.$$

- If we fix the rival players' strategies x^{-w}, the aim of the player w is to choose a strategy $x^w \in K_w(x^{-w})$ which solves the optimization problem:

$$\operatorname*{Min}_{x^w} f^w(x^w, x^{-w})$$

$$\text{such that } x^w \in K_w(x^{-w}).$$

- A tuple of N players strategies $y = (y^1, y^2, \ldots, y^{w-1}, y^w, y^{w+1}, \ldots, y^N)$ is called a **Generalized Nash equilibrium** if for each $w = 1, 2, \ldots, N$, y^w is an optimal solution of the above-mentioned optimization problem with $y^{-w} = x^{-w}$.

Notation

- There are N players; each called player $w(w = 1, 2, \dots, N)$,
- $s = (s^\alpha) \in \Omega_{s_\circ, s_1}$: multidimensional parameter of evolution, where $\alpha = 1, 2, \dots, m$ and $\Omega_{s_\circ, s_1}$ is a hyperparallelepiped with the opposite diagonal points $s_\circ = (s_\circ^1, s_\circ^2, \dots, s_\circ^m)$ and $s_1 = (s_1^1, s_1^2, \dots, s_1^m)$ in $\mathbb{R}_+^m$,
- $\Omega_{s_\circ, s_1}$: it is equivalent to the closed interval $s_\circ \leq s \leq s_1$ via the product order on $\mathbb{R}_+^m$,
- $s = (s^1, s^2, \dots, s^m)$: in noncooperative strategic games, it can be motivated as; s^1 as the given physical time, s^2 as the availability of players, s^3 as the facility given to players, s^4 as the time period choice of players, s^5 as the temporal effect (which concerns the player's strategy behavior that changes as time evolves), and so on.

- $x = x(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$: vector of strategies of all players at the given evolution parameter s, in other words strategy vector x is a function of the parameter s,
- $x^w = x^w(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^{n_w})$: vector of strategies of the player w,
- $x^{-w} = x^{-w}(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^{n-n_w})$: vector of strategies of all players except the player w,
- n: $n = \sum_{w=1}^{N} n_w$,
- $x(s) = (x^w(s), x^{-w}(s)) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$: this is just another way of writing the vector $x(s)$ in $L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$, that is,

$$x(s) = (x^1(s), x^2(s), \ldots, x^{w-1}(s), x^w(s), x^{w+1}(s), \ldots, x^N(s)),$$

- K: a nonempty, closed and convex subset of $L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$,
- $K_w(x^{-w}(s))$: nonempty, closed and convex feasible set (strategy set) in $L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^{n_w})$ of the player w for any given strategy vector $x^{-w}(s)$ of rival players,
- $x^w(s) \in K_w(x^{-w}(s))$: represents that the strategy vector $x = x(s)$ is feasible for all $w = 1, 2, \ldots, N$ and for all $s \in \Omega_{s_\circ,s_1}$,
- $F^w(x(s)) = \displaystyle\int_{\Gamma_{s_\circ,s_1}} f^w_\alpha(x^w(s), x^{-w}(s))ds^\alpha$: total cost/loss function that the player w incurs when the rival players have chosen the strategy $x^{-w}(s)$.

Glimpse of the Cost Function

$F^w : L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n) \to \mathbb{R}$ is defined in terms of the path-independent curvilinear integral

$$F^w(x(s)) = \int_{\Gamma_{s_\circ,s_1}} f^w_\alpha(x^w(s), x^{-w}(s))ds^\alpha,$$

where the summation over the repeated indices is assumed, $\Gamma_{s_\circ,s_1} \subset \Omega_{s_\circ,s_1}$ is a piecewise smooth curve joining the opposite diagonal points $s_\circ$ and s_1 in $\Omega_{s_\circ,s_1}$, $ds^\alpha = (ds^1, ds^2, \ldots, ds^m)$ is the differential element of the multidimensional parameter of evolution s^α, $f^w_\alpha(x^w(s), x^{-w}(s))$ is a real-valued continuously differentiable function, and $f^w_\alpha(x^w(s), x^{-w}(s))ds^\alpha$, is a nonautonomous closed (completely integrable) Lagrangian 1-form, that is, it satisfies

$$D_\alpha f^w_\beta = D_\beta f^w_\alpha,\ \alpha, \beta = 1, 2, \ldots, m \text{ and } \alpha \neq \beta,$$

where D_α and D_β are total derivative operators.

Formulation of MDGNEP

To find a feasible strategy $x = x(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$ such that for all $w = 1, 2, \ldots, N$, we have $x^w(s) \in K_w(x^{-w}(s))$ and for all $p^w(s) \in K_w(x^{-w}(s))$, the following inequality holds

$$F^w(x(s)) = F^w(x^w(s), x^{-w}(s)) \leq F^w(p^w(s), x^{-w}(s)).$$

Special Cases

- If the multidimensional parameter of evolution $s = (s^\alpha) \in \Omega_{s_\circ,s_1}$ is a single or linear dimensional parameter of evolution, that is, $m = 1$, then $\Omega_{s_\circ,s_1}$ is simply the closed real interval $[s_\circ, s_1]$ in $\mathbb{R}_+$ (set of non-negative real numbers). Further, we consider $\Gamma_{s_\circ,s_1} = [0, T] \subset [s_\circ, s_1]$ where T denotes an arbitrary time. Now, our formulated (MDGNEP) reduces to the time-dependent generalized Nash equilibrium problem, studied by Aussel, Gupta and Mehra (2016).

- If all the functions are independent of the multidimensional parameter of evolution s then our formulated (MDGNEP) reduces to the classical generalized Nash equilibrium problem.

Variational Inequality Formulation of MDGNEP

- The set-valued map $A : K \to 2^K$, given as

$$A(x(s)) = \prod_{w=1}^{N} K_w(x^{-w}(s)) \ \forall \, x(s) \in K.$$

- A functional $Z : K \to \mathbb{R}$ which is defined as

$$Z(x(s)) = \int_{\Gamma_{s_\circ, s_1}} z_\alpha(x(s)) ds^\alpha,$$

  where z_α is the real-valued continuously differentiable function and $z_\alpha(x(s))ds^\alpha$ is nonautonomous closed (completely integrable) Lagrangian 1-form.
- $\frac{\partial z_\alpha}{\partial x}(y(s))$ is the partial derivative of the function z_α with respect to the argument $x(s)$ at the strategy vector $y(s) \in K$.

- The multidimensional quasi-variational inequality problem is defined as follows:
  (MDQVIP) to find a vector $y(s) \in K$ such that $y(s) \in A(y(s))$ and

$$\int_{\Gamma_{s_\circ, s_1}} \left\langle \frac{\partial z_\alpha}{\partial x}(y(s)), x(s) - y(s) \right\rangle ds^\alpha \geq 0 \ \forall\, x(s) \in A(y(s)).$$

- Assume that $\frac{\partial z_\alpha}{\partial x}(x(s)) = \left(\frac{\partial f_\alpha^w}{\partial x^w}(x(s))\right)_{w=1}^N$ for any $x(s) \in K$, and for each $w = 1, 2, \ldots, N$ and each $x^{-w}(s)$, the multidimensional cost functional F^w is convex on K in the argument $x^w(s)$. Then, $y(s) \in K$ is a multidimensional generalized Nash equilibrium if and only if it is a solution to (MDQVIP).

Existence of Equilibrium

Let $y(t) \in K$ be an arbitrary strategy vector, $\frac{\partial z_\alpha}{\partial x}(y(s)) = \left(\frac{\partial f_\alpha^w}{\partial x^w}(y(s))\right)_{w=1}^N$, and for each $w = 1, 2, \ldots, N$ and a given $y^{-w}(s)$ the multidimensional cost functional F^w be convex on K in the argument $y^w(s)$. Assume that there exist a nonempty, closed and compact subset $D \subset K$ and $\hat{y}(s) \in D$ such that

$$\sum_{w=1}^{N} \int_{\Gamma_{s_\circ, s_1}} \left\langle \frac{\partial f_\alpha^w}{\partial x^w}(\hat{y}(s)), \hat{y}^w(s) - x^w(s) \right\rangle ds^\alpha < 0 \, \forall \, x(s) \in K \backslash D \text{ with } \hat{y}(s) \in A$$

Then (MDQVIP) has a solution.

Notation

- The electricity market is centralized by an independent system operator (ISO) and has N nodes.
- Each node consists of only one generator/producer $w \in M = 1, 2, \ldots, N$ and inelastic demand at each node is known,
- $s = (s^1, s^2, \ldots, s^m)$: multidimensional parameter of evolution, in electricity market problems, it can be interpreted as; s^1 as the time period for generating/delivery of electricity, s^2 as the manpower, s^3 as the generator's capacity of producing electricity, and so on,
- $D^w(s) \in L^2(\Omega_{s_\circ, s_1}, \mathbb{R}_+)$: demand function,
- $p^w(s) \in L^2(\Omega_{s_\circ, s_1}, \mathbb{R}_+)$: quantity of generated electricity (production function) by the generator w,

- $p^{-w}(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}_+^{N-1})$: quantity of generated electricity by all generators except the generator w,
- $A^w(s)p^w(s) + B^w(s)(p^w(s))^2$: true cost to generate the $p^w(s)$ units of electricity by the generator w, where $A^w(s),\ B^w(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}_+)$ are the true parameters values.

The ISO's Problem

- $a^w(s),\ b^w(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}_+)$: bid parameters,
- $a^w(s)p^w(s) + b^w(s)(p^w(s))^2$: bid generation cost (bid function) provided by w^{th} generator to ISO.
- ISO computes a production vector $p(s) = (p^w(s))_{w\in M}$ to minimize the total generation cost bidden by the all generators.

ISO has to solve the following multidimensional variational problem:

$$\text{minimize}_{p(s)} \sum_{w=1}^{N} \int_{\Gamma_{s_\circ,s_1}} (a^w(s)p^w(s) + b^w(s)(p^w(s))^2, \underbrace{\ldots}_{m \text{ times}},$$

$$a^w(s)p^w(s) + b^w(s)(p^w(s))^2)ds^\alpha$$

$$\text{subject to } \Big\{p^w(s) \geq D^w(s),\ \forall\, w \in M, \text{ a.e. on } \Gamma_{s_\circ,s_1}. \tag{1}$$

The solution set of multidimensional variational problem (1) is denoted by S_{ISO}. For the further demonstration, we define a set for the given $p^{-w}(s)$ as

$$S_w(p^{-w}(s)) = \{p^w(s) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}_+) : (p^w(s), p^{-w}(s)) \in S_{\text{ISO}}\}.$$

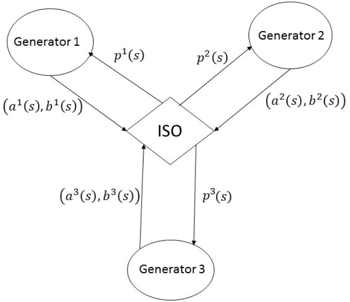

Figure: Spot Electricity Market Problem for three Generators

The Generator's Problem

- Every generator w bids a electricity generating cost $a^w(s)p^w(s) + b^w(s)(p^w(s))^2$ to the ISO and then generator w receives the best quantity of electricity $p^w(s)$ from the ISO.
- $a^w(s)+2b^w(s)p^w(s)$: market price of the generating electricity for the generator w (marginal price),
- $(a^w(s)+2b^w(s)p^w(s))p^w(s)$: revenue received by the generator w,
- $(a^w(s)p^w(s) + 2b^w(s)(p^w(s))^2) - (A^w(s)p^w(s) + B^w(s)(p^w(s))^2)$: profit of the generator w.
- We note here that the production vector $p(s)$ is supplied by the ISO problem (1).

The **multidimensional spot electricity market problem** with each generator w, is to solve the following multidimensional bilevel variational problem:

$$\text{maximize}_{a^w(s),b^w(s),p(s)} \int_{\Gamma_{s_\circ,s_1}} ([(a^w(s)p^w(s) + 2b^w(s)(p^w(s))^2)$$

$$-(A^w(s)p^w(s) + B^w(s)(p^w(s))^2)],$$

$$\underbrace{\dots}_{m \text{ times}}, [(a^w(s)p^w(s) + 2b^w(s)(p^w(s))^2)$$

$$-(A^w(s)p^w(s) + B^w(s)(p^w(s))^2)])ds^\alpha$$

$$\text{subject to} \begin{cases} a^w(s) \in [E^w(s), F^w(s)], \text{ a.e. on } \Gamma_{s_\circ,s_1}, \\ b^w(s) \in [P^w(s), Q^w(s)], \text{ a.e. on } \Gamma_{s_\circ,s_1}, \\ p(s) = (p^w(s))_{w\in M} = (p^w(s), p^{-w}(s)) \in S_{\text{ISO}}, \text{ a.e. on } \Gamma_{s_\circ,s_1}, \end{cases} \tag{2}$$

where $0 \leq E^w(s) \leq F^w(s)$ and $0 \leq P^w(s) \leq Q^w(s)$ define the feasible range for the bids $(a^w(s), b^w(s))$.

An equilibrium of the generator's problem (2) is a multidimensional generalized Nash equilibria in the sense of our formulated (MDGNEP), if we follow the following notation

$$x(s) = (a(s), b(s), p(s)),\ x^w(s) = (a^w(s), b^w(s), p^w(s)),$$

$$x^{-w}(s) = (a^{-w}(s), b^{-w}(s), p^{-w}(s)),$$

$$\begin{aligned} f_\alpha^w(x(s)) &= (f_1^w(x(s)), f_2^w(x(s)), \dots, f_m^w(x(s))) \\ &= ([(A^w(s)p^w(s) + B^w(s)(p^w(s))^2) - (a^w(s)p^w(s) + 2b^w(s)(p^w(s))^2)], \\ &\qquad \underbrace{\dots}_{\text{m times}}, [(A^w(s)p^w(s) + B^w(s)(p^w(s))^2) \\ &\qquad -(a^w(s)p^w(s) + 2b^w(s)(p^w(s))^2)]), \end{aligned}$$

$$K_w = \{(a^w(s), b^w(s)) : E^w(s) \le a^w(s) \le F^w(s) \text{ and}$$

$$P^w(s) \le b^w(s) \le Q^w(s),\ \text{a.e., on } \Gamma_{s_\circ, s_1}\},$$

$$K_w(x^{-w}(s)) = K_w \times S_w(p^{-w}(s)).$$

System of Projected Dynamical System

We consider the following system of projected dynamical system (SPDS) for $\alpha = 1, 2, \ldots, m$ on the set-valued map A, where $x(.) \in A(x(.))$:

$$\begin{aligned} \frac{dx(\cdot,\tau)}{d\tau} &= \Pi_{A(x(\cdot,\tau))}\left(x(\cdot,\tau), -\frac{\partial z_\alpha}{\partial x}(x(\cdot,\tau))\right), \\ x(\cdot,0) &= x_\circ(\cdot) \in A(x(\cdot)), \end{aligned} \tag{3}$$

where $\Pi_{A(x(\cdot))} : L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n) \times L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n) \to L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$ is the operator defined by

$$\Pi_{A(x(\cdot))}(x(.), v(.)) := \lim_{\delta \to 0^+} \frac{\mathrm{proj}_{A(x(\cdot))}(x(\cdot) + \delta v(\cdot)) - x(\cdot)}{\delta},$$

for all $x(.) \in A(x(\cdot))$ and $v(.) \in L^2(\Omega_{s_\circ,s_1}, \mathbb{R}^n)$.

Formulation of SPDS as MDGNEP

- A point $y(\cdot) \in K$ is called a critical point of (SPDS) if $y(.) \in A(y(.))$ and for $\alpha = 1, 2, \ldots, m$

$$\Pi_{A(y(\cdot))}\left(y(\cdot), -\frac{\partial z_\alpha}{\partial x}(y(\cdot))\right) = 0.$$

- Assume that $\frac{\partial z_\alpha}{\partial x}(x(.)) = \left(\frac{\partial f_\alpha^w}{\partial x^w}(x(.))\right)_{w=1}^{N}$ for any $x(.) \in K$, and for each $w = 1, 2, \ldots, N$ and each $x^{-w}(.)$, the multidimensional cost functional F^w is convex on K in the argument $x^w(.)$. Then, $y(.) \in K$ is a multidimensional generalized Nash equilibrium if and only if it is a critical point of (SPDS).

Numerical Experiments

- We solve the formulated multidimensional spot electricity market model for three generators, $w = 1, 2, 3$, see Figure 1.
- Let $m = 2$ and $\Omega_{s_\circ, s_1} = \Omega_{(1,2),(5,10)}$ be the rectangle determined by the opposite diagonal points $(1, 2)$ and $(5, 10)$ in $\mathbb{R}^2_+$.
- $\Gamma_{(1,2),(5,10)}$ be the piecewise smooth curve joining the opposite diagonal points $(1, 2)$ and $(5, 10)$ of $\Omega_{(1,2),(5,10)}$.
- We consider the parameter $s = (s^\alpha) = (t, c) \in \Omega_{(1,2),(5,10)}$ where t represents the given time period for producing electricity and c denotes the given generator's capacity of producing electricity.
- The constraint set of the ISO problem is given as

$$K_{\text{ISO}} = \{p(s) \in L^2(\Omega_{(1,2),(5,10)}, \mathbb{R}^3_+) : p^w(s) \geq D^w(s) \, \forall \, w = 1, 2, 3$$
$$\text{and a.e. on } \Gamma_{(1,2),(5,10)}\}.$$

- We discretize the $\Omega_{(1,2),(5,10)}$ as by selecting points $s_i = (t_i, c_i) \in \left\{ \left(\frac{k}{4}, \frac{k}{2} \right) : k \in \{4, 5, 6, \ldots, 20\} \right\}$.
- we get the following (SPDS) at each point s_i: for each $w = 1, 2, 3$, we have to find a point $y(s_i) = (a(s_i), b(s_i), p(s_i)) \in K$ subset of $L^2(\Omega_{(1,2),(5,10)}, \mathbb{R}_+^{3+3+3})$ such that $y^w(s_i) = (a^w(s_i), b^w(s_i), p^w(s_i)) \in K_w(s_i) \times S_w(p^{-w}(s_i))$ and for all $(\hat{a}^w(s_i), \hat{b}^w(s_i), p^w(s_i)) \in K_w(s_i) \times S_w(p^{-w}(s_i))$, the following holds

$$\int_{\Gamma_{(1,2),(5,10)}} [p^w(s_i)(a^w(s_i) - \hat{a}^w(s_i)) + 2(p^w(s_i))^2(b^w(s_i) - \hat{b}^w(s_i))](dt + dc)$$

- After a simple calculation, we find the critical points of each (SPDS) for each s_i. We then interpolate the points and finally get the approximate curves of equilibria of multidimensional spot electricity market problem (2). They are displayed in Figure 2.

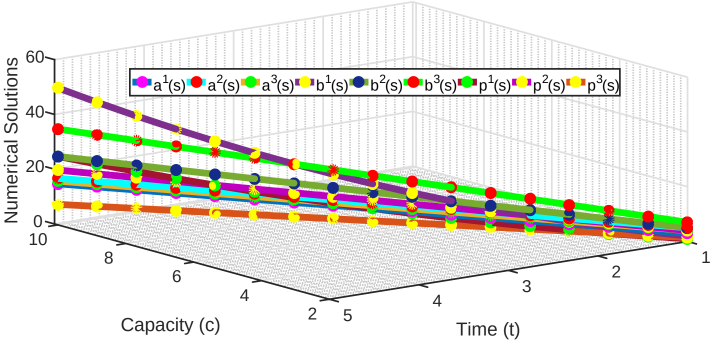

Figure: Curves of Equilibria of Multidimensional Spot Electricity Market Problem

A Comparative Study

Generator(w)	Max. Profit (MDGNEP)	Max. Profit (DGNEP)
1	381312	4762.6667
2	3041568	37770.6667
3	10255248	127104

Note: For calculating the maximize profit of each generator in the case of (MDGNEP), we consider the piecewise smooth curve $\Gamma_{(1,2)(5,10)}$ as $(t, 2t)$, that is $c = 2t$, for $1 \leq t \leq 5$.

THANK YOU