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Extragradient-Type Algorithm for Solving Pseudomonotone Equilibrium Problem with Bregman Distance in Reflexive Banach Spaces

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(d) Application to GNEP in differential games.

(e) Numerical illustrations given to support the theoretical analysis.

Introduction

E = reflexive Banach space; $C \neq \emptyset$, C = closed, convex subset of E; $g : E \times E \to \mathbb{R}$ such that $g(x,x) = 0, \forall x \in E$. In this talk, we consider the Equilibrium Problem (shortly EP):

$$\text{find} \quad z^* \in C \quad \text{such that} \quad g(z^*, y) \geq 0, \quad \forall\, y \in C. \tag{1}$$

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Iusem and Sosa (2003) showed that the generalization given by the EP (1) formulation above is genuine, in the sense that there are EP problems which do not fit the format of such particular cases mentioned above.

Definition

A bifunction $g : E \times E \to \mathbb{R}$ is said to be

monotone if $g(x, y) + g(y, x) \leq 0, \ \forall x, y \in E$;

pseudo-monotone if $g(x, y) \geq 0 \Longrightarrow g(y, x) \leq 0, \ \forall x, y \in E$.

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Definition

The bifunction $g : E \times E \to \mathbb{R}$ is said to satisfy a Lipschitz-type condition (see Mastroeni 2003) if there exist two positive constants c_1, c_2 such that

$$g(x, y) + g(y, z) \geq g(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \quad \forall x, y, z \in E. \tag{2}$$

Definition

Bregman (1967) Given $f : E \to \mathbb{R}$, a strictly convex and Gâteaux differentiable function. The Bregman distance with respect to f is the function $D_f : \text{dom} f \times \text{int}(\text{dom} f) \to [0, +\infty)$ defined by

$$D_f(u, v) = f(u) - f(v) - \langle \nabla f(v), u - v \rangle \quad \forall u \in \text{dom} f, v \in \text{int}(\text{dom} f). \tag{3}$$

The Bregman distance does not satisfy all the properties of metric. The symmetric and triangular inequality properties do not hold but it has the following important properties which follows directly from its definition:

(i) $D_f(u, v) = 0$ if and only if $u = v$;

(ii) (three point identity): for any $z \in \text{dom} f$ and $u, v \in \text{int}(\text{dom} f)$

$$\langle \nabla f(v) - \nabla f(z), z - u \rangle = D_f(u, v) - D_f(u, z) - D_f(z, v); \tag{4}$$

Iterations

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In 2008, Quoc et al. (2008) extended EM to solve the pseudo-monotone EP (1) in Euclidean spaces.

Algorithm

Given $x_n \in C$ and $\lambda_n > 0$, compute y_n and x_{n+1} as follows:

$$\begin{cases} y_n = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda_n g(x_n, y) + \frac{1}{2}||y - x_n||^2 \right\} \\ x_{n+1} = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda_n g(y_n, y) + \frac{1}{2}||y - x_n||^2 \right\}, \end{cases} \quad (5)$$

where $0 < \lambda_n < \min\left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\}$.

2. EM has further been extended to infinite dimensional settings. E.g., Anh and An 2015; 2019; Bigi et al. 2009; Bigi and Passacantando 2015; Hieu 2018a; Hieu et al. 2018; Jolaoso and Aphane 2020a; ur Rehman et al. 2019; Vuong et al. 2013; Vuong 2018.

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EM involves solving minimization problem on C twice per iteration and two evaluations of $g(.,y)$ per iteration. This could be computationally expensive and thus, a drawback.

3. Lyashko and Semenov (2016), proposed a Popov's type extragradient algorithm to solve EP (1). The method involves two minimization problems on C but one evaluation of $g(., y)$ per iteration in Hilbert spaces.

Algorithm

Given $x_n, y_n \in C$, compute x_{n+1} and y_{n+1} as follows:

$$\begin{cases} x_{n+1} = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda g(y_n, y) + \frac{1}{2}||y - x_n||^2 \right\} \\ y_{n+1} = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda g(y_n, y) + \frac{1}{2}||y - x_{n+1}||^2 \right\}, \end{cases} \tag{6}$$

where $0 < \lambda < \frac{1}{2(c_1+c_2)}$.

Example

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Let $E = \ell^p := \{\zeta = (\zeta_1, \zeta_2, \dots) : (\sum_{i=1}^{\infty} |\zeta_i|^p)^{\frac{1}{p}} < \infty\}$ for $1 < p < \infty$ and $f : E \to \mathbb{R}$ be defined by $f(x) = \|x\|^p$. Let

$$C = \{\zeta : (\zeta_1, \zeta_2, \dots) \in \ell^p : \zeta_i \geq 0 \text{ and } \|\zeta\| \leq p, \quad \forall i \in \mathbb{N}\}$$

and $g : C \times C \to \mathbb{R}$ be defined by $g(x, y) = (p - ||x||)\langle x, y - x\rangle$ for all $x, y \in C$. Clearly, $EP(g) \neq \emptyset$, g is pseudo-monotone and not monotone ($x = (\frac{5}{2}, 0, 0, \dots)$ and $y = (3, 0, 0, \dots)$). Also, g satisfies Lipschitz-like condition with $c_1 = c_2 = \frac{3p}{2}$. However, all the above-mentioned methods cannot be applied to solve the EP (1).

Some other methods have been proposed to solve EP in reflexive Banach spaces, e.g., Takahashi & Zembayashi (2009), Reich & Sabach (2010), Kassay et al. (2011), Eskandani et al. (2020), etc. Most of the methods proposed either involve two evaluations of $g(., y)$ per iteration or minimization problem on C twice per iteration or projection onto intersection of two half-spaces.

Our Method

Algorithm

Given x_n, y_{n-1}, y_n and $\lambda_n > 0$, compute x_{n+1} and y_{n+1} as follows:

$$\begin{cases} x_{n+1} = \underset{y \in H_n}{\operatorname{argmin}} \Big\{ \lambda_n g(y_n, y) + D_f(y, x_n) \Big\} \\ y_{n+1} = \underset{y \in C}{\operatorname{argmin}} \left\{ \lambda_{n+1} g(y_n, y) + D_f(y, x_{n+1}) \right\}, \end{cases} \tag{7}$$

Remark

1. The method involves one minimization problem on C and one evaluation of $g(., y)$ per iteration with self-adaptive step sizes λ_n.

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1. The method involves one minimization problem on C and one evaluation of $g(.,y)$ per iteration with self-adaptive step sizes λ_n. 2. The method is proposed in a reflexive Banach space which is more general than most available methods.

We assume that the following assumptions are satisfied.

Condition

(B1) f is proper, convex and lower semicontinuous;

(B2) f is uniformly Fréchet differentiable;

(B3) f is β-strongly convex on every $C \subset E$;

(B4) f is a strongly coercive and Legendre function which is bounded.

Condition

Let $g : E \times E \to \mathbb{R}$ be a bifunction such that

(A1) g is pseudomonotone;

(A2) g satisfies the Lipschitz-like condition;

(A3) $g(x, \cdot)$ is convex, lower semicontinuous and subdifferentiable for every $x \in C$;

(A4) $g(\cdot, y)$ is sequentially weakly upper semicontinuous on C for every fixed $y \in C$, i.e., if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x \in C$, then $\limsup_{n\to\infty} g(x_n, y) \leq g(x, y)$;

(A5) For all bounded sequences $\{x_n\}, \{y_n\} \in C$ such that $\|x_n - y_n\| \to 0, n \to \infty$, the inequality

$$\limsup_{n\to\infty} g(x_n, y_n) \geq 0$$

holds;

(A6) $EP(g) \neq \emptyset$.

Let $A : C \to E^*$ be an operator and $g(x, y) := \langle Ax, y - x \rangle$ for every $x, y \in C$, the EP reduces to VIP(C,A):

$$\text{find} \quad x^* \in C \quad \text{such that} \quad \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall\, y \in C. \tag{8}$$

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then Conditions (A1)-(A6) are satisfied with $L = \frac{c_1}{2} = \frac{c_2}{2}$.

Algorithm

Given x_n, y_{n-1}, y_n and $\lambda_n > 0$, compute x_{n+1} and y_{n+1} as follows:

$$\begin{cases} x_{n+1} = Proj^f_{H_n}\Big(\nabla f^*[\nabla f(x_n) - \lambda_n A(y_n)]\Big) \\ y_{n+1} = Proj^f_C\Big(\nabla f^*[\nabla f(x_{n+1}) - \lambda_{n+1} A(y_n)]\Big), \end{cases} \tag{9}$$

Lemma

Suppose $\{x_n\}$ and $\{y_n\}$ are generated by our proposed Algorithm. Then

$$\begin{aligned} D_f(p, x_{n+1}) &\leq D_f(p, x_n) - \left(1 - \frac{2\sigma\lambda_n}{\beta\lambda_{n+1}}\right) D_f(y_n, x_n) \\ &\quad - \left(1 - \frac{\sigma\lambda_n}{\beta\lambda_{n+1}}\right) D_f(x_{n+1}, y_n) + \frac{2\sigma\lambda_n}{\beta\lambda_{n+1}} D_f(x_n, y_{n-1}), \end{aligned}$$

for all $n \geq 1$, where $p \in EP(g)$, β is the strongly convexity constant of f and $\sigma \in (0, \beta/3)$.

Theorem

Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Suppose that f, g satisfy above Conditions. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (7) converge weakly to a solution of the EP (1).

Corollary

Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Assume that A satisfies the above conditions. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (9) converges weakly to a solution of the $VIP(C, A)$.

Application

Let $I = \{1, 2, \ldots, N\}$ be the set of players with each player $i \in I$ controlling variable $x^i \in \mathcal{B}_i$, $\mathcal{B}_i$ is a Banach space and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \times \mathcal{B}_N$. The point x^i is called the strategy of the ith player. We denote by $x \in \mathcal{B}$, the vector of strategies $x = (x_1, \ldots, x_N)$ and x^{-i} denotes the vector formed by all player decision variables x^j except the player i. Thus, we can write $x = (x^i, x^{-i})$ which is the shorthand to denote the vector $x = \left(x^1, \ldots, x^{i-1}, x^i, x^{i+1}, \ldots, x^N\right)$. Given a subset $\mathcal{X}$ of $\mathcal{B}$ (called the feasible set for the GNEP), the set $\mathcal{X}_i(x^{-i}) = \{x^i \in \mathcal{B}_i : (x^i, x^{-i}) \in \mathcal{X}\}$ denotes the strategy set of the ith player when the remaining player choose strategies x^{-i} (see, e.g., Rosen (1965)). We note that the aim of the ith player given the strategy x^{-i} is to choose a strategy x^i such that x^i solves the following minimization problem:

$$\min \theta_i(x^i, x^{-i}) \quad \text{such that} \quad x^i \in \mathcal{X}_i(x^{-i}). \tag{10}$$

For any given x^{-i}, we denote the solution set of (10) by $Sol_i(x^{-i})$. Using the above notation, we give the precise definition of the GNEP as follows (see, e.g., Iusem and Nasri (2011)).

Definition

A GNEP is define as finding $\bar{x} \in \mathcal{X}$ such that $\bar{x}^i \in Sol_i(\bar{x}^{-i})$ for every $i \in I$.

Theorem

Consider the GNEP such that

(a) $\mathcal{X}$ *is closed and convex,*

(b) θ_i *is continuously differentiable for every* $i \in I$,

(c) $\theta_i(\cdot, x^{-i}) : \mathcal{B}_i \to \mathbb{R}$ *is convex for every* $i \in I$ *and every* $x \in \mathcal{X}$.

Define an operator $F : \mathcal{B} \to \mathcal{B}^*$ *as*

$$F(x) = \Big(\nabla_{x^1}\theta_1(x), \ldots, \nabla_{x^N}\theta_N(x)\Big),$$

where $\nabla_{x^i}\theta_i$ *denotes the gradient of* θ_i *with respect to its first argument. Then every solution of* $EP(g)$, *with* $g(x, y) := \langle F(x), y - x\rangle$ *is a solution of the GNEP.*

Example

Consider the EP (1) with $g : C \times C \to \mathbb{R}$ defined by

$$g(x,y) = \langle \mathcal{M}x + \mathcal{F}(y) + q, y - x \rangle$$

where q is a vector in $\mathbb{R}^m$, $\mathcal{M}$ and $\mathcal{F}$ are $m \times m$ matrices such that $\mathcal{M}$ is symmetric and positive semidefinite, and $\mathcal{F} - \mathcal{M}$ is negative semidefinite. The feasible set $C \subset \mathbb{R}^m$ is defined by $C = \{x \in \mathbb{R}^m : Qx \leq b\}$, where Q is a matrix of size $l \times m$ generated randomly in $[-2,2]$ and b is a vector in $\mathbb{R}^m$ generated randomly in [1,3]. We choose the Bregman function $f(x) = \frac{1}{2}||x||^2$.

table

Table: Computation result

		Prop.Alg.	EGM	SEM	PEM
$m = 5$	Iter.	23	35	54	31
	Time (sec)	0.5351	0.9021	1.6107	0.6605
$m = 10$	Iter.	68	98	132	90
	Time (sec)	3.5926	3.9732	4.9471	3.7579
$m = 20$	Iter.	166	237	274	222
	Time (sec)	3.8769	5.2072	6.2164	4.7159
$m = 50$	Iter.	292	414	449	389
	Time (sec)	7.3758	13.2904	16.2664	8.1766

Example

Consider the same bifunction g defined in previous example and

$$C := \Big\{x = (x_1, \dots, x_m)^T : ||x|| \leq 1 \quad \text{and} \quad x_i \geq a,\ i = 1, \dots, m\Big\}$$

where $a < 1/\sqrt{m}$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$(i) \quad f(x) = \sum_{i=1}^{m} x_i \log(x_i), \quad (ii) \quad f(x) = -\sum_{i=1}^{m} \log(x_i) \quad (iii) \quad f(x) = \frac{1}{2}||x||^2.$$

Then

$$(i) \quad \nabla f(x) = \Big(1 + \log(x_1), \dots, 1 + \log(x_m)\Big)^T$$

$$(ii) \quad \nabla f(x) = -\Big(\frac{1}{x}, \dots, \frac{1}{x_m}\Big)^T$$

$$(iii) \quad \nabla f(x) = x,$$

respectively.

Example

Moreover,

(i) $D_f(x,y) = \sum_{i=1}^{m} \left(x_i \log\left(\frac{x_i}{y_i}\right) + y_i - x_i \right)$ which is the Kullback-Leibler distance (KLD),

(ii) $D_f(x,y) = \sum_{i=1}^{m} \left(\frac{x_i}{y_i} - \log\left(\frac{x_i}{y_i}\right) - 1 \right)$ which is the Itakura-Saito distance (ISD),

(iii) $D_f(x,y) = \frac{1}{2}||x-y||^2$ which is the squared Euclidean distance (SED), table

Table: Computation result

		KLD	ISD	SED
$m = 10$	Iter.	44	30	37
	Time (sec)	0.8881	0.9880	1.7544
$m = 50$	Iter.	357	240	176
	Time (sec)	23.1684	14.5520	10.3720
$m = 70$	Iter.	546	365	270
	Time (sec)	39.1003	25.6851	20.0191
$m = 100$	Iter.	869	581	432
	Time (sec)	75.6700	49.9736	35.5233

Future Research

Inertial version of our proposed method in reflexive Banach spaces has not been studied in the literature.

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A modification of our method without any further minimization on the half-space H_n would be desired.

THANKS FOR LISTENING

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