Embedding of partially ordered topological spaces in Fell topological hyperspaces

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Abstract

Let $(X, \tau, \preccurlyeq)$ be a partially ordered topological space. For every $x \in X$, we denote the subset of X $x^{\downarrow}=\{u \in X: u \preccurlyeq x\}$. Let $C(X)$ denote the collection of all τ-closed subsets of X and $C^{\downarrow}(X) = \{x^{\downarrow}: x \in X\}$. We consider three ways to topology the hyperspace $C(X)$:

(i) The Fell topology τ_F on $C(X)$, which has a base consisting of the following elements

$$O^{-} = \{A \in C(X): A \cap O \neq \emptyset\} \quad \text{and} \quad (X \backslash D)^{+} = \{A \in C(X): A \cap D = \emptyset\},$$

for every τ-open subset O of X and every τ-compact subset D of X;

(ii) The Vietoris topology τ_F on $C(X)$, which has a base consisting of O^{-} and $(X \backslash D)^{+}$ as defined in (i) with every τ-closed subset D of X;

(iii) The Hausdorff topology τ_H on $C(X)$ for (X, τ) being a metric space and the topology τ is induced by a metric d on X. The Hausdorff metric H on $C(X)$ is defined, for any distinct $A, B \in C(X)$, as

$$H(A, B) = \max\left\{\sup_{a \in A}\left(\inf_{b \in B} d(a, b)\right), \sup_{b \in B}\left(\inf_{a \in A} d(b, a)\right)\right\}.$$

In this paper, we consider some properties of the canonical map $x \to x^{\downarrow}$ which is from X to $C(X)$ with respect to the above three topologies τ_F, τ_V, and τ_H on $C(X)$. We first prove a result that the canonical map topologically order-embeds $(X, \tau, \preccurlyeq)$ in $(C^{\downarrow}(X), \tau_F, \subseteq)$, in which the canonical map $x \to x^{\downarrow}$ satisfies the following conditions:

E_1. $x \preccurlyeq y$ if and only if $x^{\downarrow} \subseteq y^{\downarrow}$;
E_2. $x = y$ if and only if $x^{\downarrow} = y^{\downarrow}$;
E_3. The canonical map $x \to x^{\downarrow}$ is continuous from (X, τ) to $(C^{\downarrow}(X), \tau_F)$;
E_4. The map $x^{\downarrow} \to x$ is continuous from $(C^{\downarrow}(X), \tau_F)$ to (X, τ).

Then, we give some counterexamples of locally compact and order-connected (topological) metric $\wedge$-semilattices $(X, \tau, \preccurlyeq)$ (that are special cases of partially ordered metric spaces), in which the canonical map $x \to x^{\downarrow}$ has the following properties:

(a) It topologically order-embeds $(X, \tau, \preccurlyeq)$ in $(C^{\downarrow}(X), \tau_F, \subseteq)$;
(b) It is not continuous at every point (excepting one point) from (X, τ) to $(C^{\downarrow}(X), \tau_V)$;
(c) It is not continuous at every point from (X, τ) to $(C^{\downarrow}(X), \tau_H)$.