Embedding of partially ordered topological spaces in Fell topological hyperspaces

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Abstract

Let \((X, \tau, \leq)\) be a partially ordered topological space. For every \(x \in X\), we denote the subset of \(X\)
\[ x^\downarrow = \{ u \in X : u \leq x \}. \]
Let \(C(X)\) denote the collection of all \(\tau\)-closed subsets of \(X\) and \(C^\downarrow(X) = \{ x^\downarrow : x \in X \}\). We consider three ways to topology the hyperspace \(C(X)\):

(i) The Fell topology \(\tau_F\) on \(C(X)\), which has a base consisting of the following elements

\[
O^- = \{ A \in C(X) : A \cap O \neq \emptyset \} \quad \text{and} \quad (X \setminus D)^+ = \{ A \in C(X) : A \cap D = \emptyset \},
\]
for every \(\tau\)-open subset \(O\) of \(X\) and every \(\tau\)-compact subset \(D\) of \(X\);

(ii) The Vietoris topology \(\tau_V\) on \(C(X)\), which has a base consisting of \(O^-\) and \((X \setminus D)^+\) as defined in (i) with every \(\tau\)-closed subset \(D\) of \(X\);

(iii) The Hausdorff topology \(\tau_H\) on \(C(X)\) for \((X, \tau)\) being a metric space and the topology \(\tau\) is induced by a metric \(d\) on \(X\). The Hausdorff metric \(H\) on \(C(X)\) is defined, for any distinct \(A, B \in C(X)\), as

\[
H(A, B) = \max \left\{ \sup_{a \in A} \left( \inf_{b \in B} d(a, b) \right), \sup_{b \in B} \left( \inf_{a \in A} d(b, a) \right) \right\}.
\]

In this paper, we consider some properties of the canonical map \(x \to x^\downarrow\) which is from \(X\) to \(C(X)\) with respect to the above three topologies \(\tau_F, \tau_V, \text{ and } \tau_H\) on \(C(X)\). We first prove a result that the canonical map topologically order-embeds \((X, \tau, \leq)\) in \((C^\downarrow(X), \tau_F, \leq)\), in which the canonical map \(x \to x^\downarrow\) satisfies the following conditions:

- **E1.** \(x \leq y\) if and only if \(x^\downarrow \subseteq y^\downarrow\);
- **E2.** \(x = y\) if and only if \(x^\downarrow = y^\downarrow\);
- **E3.** The canonical map \(x \to x^\downarrow\) is continuous from \((X, \tau)\) to \((C^\downarrow(X), \tau_F)\);
- **E4.** The map \(x^\downarrow \to x\) is continuous from \((C^\downarrow(X), \tau_F)\) to \((X, \tau)\).

Then, we give some counterexamples of locally compact and order-connected (topological) metric \(\Lambda\)-semilattices \((X, \tau, \leq)\) (that are special cases of partially ordered metric spaces), in which the canonical map \(x \to x^\downarrow\) has the following properties:

- (a) It topologically order-embeds \((X, \tau, \leq)\) in \((C^\downarrow(X), \tau_F, \leq)\);
- (b) It is not continuous at every point (excepting one point) from \((X, \tau)\) to \((C^\downarrow(X), \tau_V)\);
- (c) It is not continuous at every point from \((X, \tau)\) to \((C^\downarrow(X), \tau_H)\).