Hidden Positivity and a New Approach to Rigorous Computation of Hausdorff Dimension: Higher Order Methods

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We shall begin with an old motivating example: the question of rigorous computation of the Hausdorff dimension of some subsets of the real numbers which can be represented by special classes of continued fractions. We shall then generalize and describe a new approach to high order rigorous approximation of the Hausdorff dimension of the invariant set of an iterated function system or IFS which acts on a subset of the reals. We extend our earlier methods by allowing higher order numerical methods, but the extension requires new ideas. We again rely on the fact that associated to our IFS is a parametrized family L(s), s>=0, of bounded positive linear operators . In our setting L(s) is not compact but has essential spectral radius strictly less than its spectral radius R(L(s)) and for every positive integer k has a unique strictly positive k-times continuously differentiable eigenfunction v(s) with eigenvalue R(L(s)). In our setting it is always true that there exists a unique s*>=0 such that R(L(s*))=1. The Hausdorff dimension h of the invariant set of the IFS always satisfies h<=s*, and under appropriate assumptions, h=s*. For a positive integer m let L(s;m) denote the mth iterate of L(s) and note that the eigenvalue problem of estimating R((L(s)) is equivalent to the problem of estimating R(L(s;m)). This eigenvalue problem is then approximated using a collocation method, extended Chebyshev points and continuous piecewise polynomials of degree r to obtain a large square matrix M(s;m) whose spectral radius R(M(s;m)) closely approximates R(L(s;m)). Using the theory of positive linear operators and explicit (s-dependent) bounds on the derivatives of the eigenfunction v(s), we obtain explicit upper and lower bounds for s*; and these upper and lower bounds converge rapidly to s* as a certain mesh size decreases.