The influence of the work of R.E. Bruck on the development of proof mining

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Proof mining is the project of applying proof-theoretic transformations to obtain new quantitative and qualitative information from given proofs in areas of core mathematics such as nonlinear analysis, convex optimization and geodesic geometry (see e.g. [10]). Bruck, who himself did fundamental work on quantitative issues of metric fixed point theory ([2]), was a major source of inspiration in this program both by providing in his research deep results which naturally asked for a more finitary quantitative treatment as well as by introducing fundamental new notions which were particularly suited for such a proof-theoretic enterprise. Examples of the former are Bruck's prima facie nonconstructive proof of the existence of sunny nonexpansive retractions ([3]), his work on the convex approximation property ([5]) and his convergence results on an iteration scheme for pseudo-contractions ([4]) which prompted proof-theoretic quantitative versions in [14] (via Reich's fundamental paper [17]), [8] and [15] respectively. Examples for the latter are his notions of firmly nonexpansive mappings ([3]), averaged and strongly nonexpansive mappings (with Reich in [7]) and strongly quasi-nonexpansive mappings ([6]) which beautifully fit the requirements for proof-theoretically well-behaved classes of nonlinear mappings and play a crucial rule in papers using the proof mining methodology such as [1, 16, 9]. In particular, the concept of strong nonexpansivity led to the proof-theoretically motivated quantitative notion of 'modulus of strong nonexpansivity' which plays a crucial role in the extraction of a polynomial rate of asymptotic regularity in [11] for Bauschke's solution of the zero displacement conjecture (see also the recent generalization of [11] in [18]) as well as the quantitative analysis of proximal point type algorithms in [12, 13].
In this talk we will survey some of these developments.

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