Optimal error bounds in fixed-point iterations for non-expansive maps via optimal transport metrics

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Abstract

We discuss optimal error bounds and convergence rates of fixed-point iterations for non-expansive maps. Building upon far-reaching ideas by Baillon and Bruck in their study of Krasnoselsk'ii-Mann iterates (1992, 1996), we look for general Mann's iterations that attain the smallest fixed-point residual after n steps. This is achieved by optimizing a worst-case bound $\|x^n - Tx^n\| \leq R_n$ derived from a nested family of optimal transport problems. We prove that this bound is tight so that minimizing R_n yields optimal iterations. Inspired from numerical results we identify iterations that attain the rate $R_n = O(1/n)$, which is proven to be the best possible. In particular, we show that the classical Halpern iteration achieves this optimal rate for several alternative stepsizes, and we determine analytically the optimal stepsizes that attain the smallest worst-case residuals at every step n, with a tight bound $R_n \approx \frac{4}{n+4}$. We also show that the best rate for the classical Krasnosel'skiĭ-Mann iteration is only $\Theta(1/\sqrt{n})$. If time permits, we will briefly describe the metric properties of the optimal transport bounds and some of their very special properties, including the so-called *convex quadrangle inequality* that yields a greedy method to compute these bounds efficiently.

This talk summarizes recent work done in collaboration with Mario Bravo, Thierry Champion, Juan Pablo Contreras, Matías Pavez-Signé, and José Soto.