Tikhonov's regularization and iterative schemes for solving split feasibility and fixed point problems in Hilbert space

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Overview

Abstract

We study the split feasibility and fixed point problems for Lipschitzian pseudocontractive and nonexpansive mappings in real Hilbert spaces. Using Tikhonov's regularization technique, we propose and analyze iterative schemes for approximating solutions to such problems.

Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let K be a nonempty, closed and convex subset of H.

- weak conv.: '$x_n \rightharpoonup x$' i.e $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$ for all $y \in H$.
- Let $T : H \to H$ be a mapping. $Fix(T) := \{x \in H : Tx = x\}$.

Definition: A mapping $T : H \to H$ is said to be:

(i) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \ \ \forall x, y \in H.$$

T is a contraction if $L \in [0, 1)$. If $L = 1$, T is nonexpansive;

(ii) Pseudocontractive if

$$\langle Tu - Tv, u - v \rangle \leq \|u - v\|^2 \ \ \forall u, v \in H;$$

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Definition

(iii) Quasi-pseudocontractive if $Fix(T) \neq \emptyset$ and

$$\langle Tu - v, u - v \rangle \leq \|u - v\|^2 \quad \forall u \in H, v \in Fix(T);$$

(iv) Monotone if

$$\langle Tu - Tv, u - v \rangle \geq 0 \; \forall u, v \in H.$$

- T is monotone if and only if $I - T$ is pseudocontractive.
- The solutions of the operator equation $Tu = 0$ coincide with the fixed points of $I - T$.

We denote the solution set of the operator equation $Tu = 0$ by zer T.

Definition

The metric projection of $x \in H$ on K is defined as

$$P_K x := \arg\min\{\|x - y\| : y \in K\}.$$

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Let H_1 and H_2 be real Hilbert spaces, $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex and $A : H_1 \to H_2$ be a bounded linear operator.

Split Feasibility Problem (Censor & Elfving, 1994)

Find x$\in$ C such that $Ax \in Q$. $\quad (SFP)$

- An iterative scheme derived from the multiprojection algorithm in (Censor & Elfving, 1994).
- The multidistance projection algorithm in (Byrne, 2001).
- CQ-algorithm (Byrne, 2002)

$$x_{n+1} = P_C(x_n - \gamma A^T(I - P_Q)Ax_n), \ n \in \mathbb{N}. \tag{1}$$

The SFP is equivalent to the following constrained optimization problem:

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_QAx\|^2. \tag{2}$$

f is continuously differentiable and its gradient ∇f is given by

$$\nabla f(x) = A^*(I - P_Q)Ax.$$

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(Xu, 2010)

$$x_1 \in H_1,\ x_{n+1} = P_C(x_n - \gamma \nabla f(x_n)),\ n \in \mathbb{N}, \tag{3}$$

where $0 < \gamma < \frac{2}{\|A\|^2}$. (3) converges weakly to a solution of the SFP.

Xu (Xu, 2010) considered the following Tikhonov regularization:

$$\min_{x \in C} f^{\tau}(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\tau\|x\|^2, \tag{4}$$

where $\tau > 0$ is the regularization parameter. Note that, $\nabla f^{\tau}(x) = A^*(I - P_Q)Ax + \tau x$.

Theorem (Xu, 2010)

$$x_1 \in H_1,\ x_{n+1} = P_C(x_n - \gamma \nabla f^{\tau_n}(x_n)),\ n \in \mathbb{N}, \tag{5}$$

where $\gamma \in (0, \frac{2}{\|A\|^2})$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. Then $\{x_n\}$ converges weakly to a solution of the SFP.

Introduction: Split Feasibility and Fixed Point Problem

Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two mappings with nonempty fixed point sets $Fix(T)$ and $Fix(S)$, respectively.

Split Common Fixed Point Problem (SCFPP) (Censor & Segal 2010)

$$\text{Find } x \in Fix(T) \text{ such that } Ax \in Fix(S). \qquad (SCFPP).$$

The SCFPP generalizes the SFP. A more general problem is the following composite problem:

Split Feasibility and Fixed Point Problem (SFFPP)

$$\text{Find } x^* \in C \cap Fix(T) \text{ such that } Ax^* \in Q \cap Fix(S). \tag{6}$$

We denote the solution set of the SFFPP by Γ.

Iterative schemes for solving SFFPP

Ceng et al.; 2012

$$\begin{cases} x_1 \in C, \\ z_n = P_C(I - \gamma_n \nabla f^{\tau_n})x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) T P_C(x_n - \gamma_n \nabla f^{\tau_n}(z_n)), \quad n \in \mathbb{N}. \end{cases} \tag{7}$$

Chen et al.; 2015

$$\begin{cases} x_1 \in C, \\ q_n = P_C(I - \gamma_n \nabla f^{S})x_n, \\ w_n = P_C(x_n - \gamma_n \nabla f^{S}(q_n)) \\ z_n = (1-\beta_n) w_n + \beta_n T w_n \\ x_{n+1} = (1-\alpha_n) w_n + \alpha_n T z_n, n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\nabla f^S := A^*(I - SP_Q)A$.

S is a nonexapnsive mapping and T is Lipschitzian pseudocontractive mapping.

Iterative schemes for solving SFFPP

Wongsasinchai, 2021

$$
\begin{cases}
x_1 \in C, \\
q_n = P_C(I - \gamma_n \nabla f^{S\tau_n})x_n, \\
w_n = P_C(x_n - \gamma_n \nabla f^{S\tau_n} q_n), \\
s_n = (1-\delta_n)w_n + \delta_n T w_n, \\
z_n = (1-\beta_n)s_n + \beta_n T s_n, \\
x_{n+1} = (1-\alpha_n)z_n + \alpha_n T z_n, n \in \mathbb{N},
\end{cases} \tag{9}
$$

where

$$\nabla f^{S\tau_n} := A^*(I - SP_Q)A + \tau_n I.$$

The sequences generated by (9) converge weakly to a point in the solution set Γ if $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\tau_n + 2\|A\|^2})$, $\{\tau_n\} \subset (0, \infty)$, $\sum_{n=1}^{\infty} \tau_n < \infty$ and $0 < a < \alpha_n < b < \beta_n < c < \delta_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$.

Motivation for the study

- The extragradient algorithms (7) - (9) for solving the SFFPP involve the computation of four metric projections per iteration. Since computing metric projection amounts to solving a minimization problem, the computation of four metric projections per iteration does increase the computational burden of the algorithm.
- Thus a more economical approach is to reduce the number of projections per iteration.
- Using Tikhonov's regularization technique, we propose and analyze iterative scheme for approximating solutions to the SFFPP for the case where S is nonexpansive and T is Lipschitzian pseudocontractive.

Preliminaries

Lemma 1

If $T : H \to H$ is a pseudocontractive mapping, then its fixed point set $Fix(T)$ is closed and convex.

Definition

Let $T : H \to H$ be a mapping. The mapping $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\}$ in H, the assumptions $x_n \rightharpoonup x^*$ and $(I - T)x_n \to 0$ imply that $Tx^* = x^*$.

Lemma 2 (Zhou, 2009)

Let H be a real Hilbert space and K be a closed and convex subset of H. Let $T : K \to K$ be a continuous pseudocontractive mapping. Then $I - T$ is demiclosed at zero.

Preliminaries

Lemma 3 (Chen et al.; 2015)

Let K be a nonempty, closed and convex subset of a real Hilbert space H and let $S : K \to K$ be a nonexpansive mapping. Set $\nabla f^S := A^*(I - SP_K)A$. Then

$$\langle x - y, \nabla f^S(x) - \nabla f^S(y)\rangle \geq \frac{1}{2\|A\|^2}\|\nabla f^S(x) - \nabla f^S(y)\|^2. \qquad (10)$$

Lemma 4 (Tan & Xu, 1993)

Let $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ be two sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n \quad \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Preliminaries

Lemma 5 (Yao et al.; 2015)

Let H be a real Hilbert space and $T : H \to H$ be an L-Lipschitzian mapping with $L \geq 1$. Set $T_\alpha = (1-\alpha)I + \alpha T((1-\kappa)I + \kappa T)$. If $0 < \alpha < \kappa < \frac{1}{1+\sqrt{1+L^2}}$ and T is quasi-pseudocontractive, then T_α is quasi-nonexpansive and

$$\|T_\alpha x - x^*\|^2 \leq \|x - x^*\|^2 - \alpha(\kappa - \alpha)(1-\kappa L)^2 \|Tx - x\|^2 \; \forall x \in H, x^* \in Fix(T).$$

Lemma 6 (Xu, 2010)

Let K be a nonempty, closed and convex subset of a Hilbert space H. Let $\{x_n\}$ be a bounded sequence which satisfies the following two properties:

• every weak limit point of $\{x_n\}$ lies in K;

and

• $\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in K$.

Then $\{x_n\}$ converges weakly to a point in K.

Main Results

Algorithm

Let C and Q be nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of A. Let $S : H_2 \to H_2$ be a nonexpansive mapping and $T : H_1 \to H_1$ be an L-Lipschitzian pseudocontractive mapping with $L \geq 1$. Suppose that $\Gamma := \{x^* \in H_1 : x^* \in C \cap Fix(T),\ Ax^* \in Q \cap Fix(S)\} \neq \emptyset$ and let $\{x_n\}$ be a sequence generated as follows:

$$\begin{cases} x_1 \in C, \\ u_n = P_C(x_n - \gamma_n \nabla f^{S\tau_n}(x_n)), \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T((1-\beta_n)u_n + \beta_n Tu_n), \quad n \in \mathbb{N}, \end{cases} \tag{11}$$

where

$$\nabla f^{S\tau_n}(x_n) = \ \nabla f^S(x_n) + \tau_n x_n,\ \nabla f^S(x_n) = A^*(I - SP_Q)Ax_n.$$

Main Result

We further assume the following conditions:

(i) $\tau_n \in [0,1)$ with $\sum_{n=1}^{\infty} \tau_n < \infty$;

(ii) $0 < a \leq \gamma_n \leq b < \frac{1}{\tau_n + \|A\|^2}$;

(iii) $0 < c < \alpha_n < d < \beta_n < e < \frac{1}{1+\sqrt{1+L^2}}$.

The following lemmata are used in the proof of our main theorem.

Lemma 3.1

Given the data in Algorithm (11), the mapping $P_C(I - \gamma_n \nabla f^{S\tau_n}) : C \to C$ is a strict contraction with constant $(1 - \gamma_n \tau_n)$.

Proof

Let $x, y \in C$. Then

$$\begin{aligned} \|P_C(x - \gamma_n \nabla f^{S\tau_n} x) - P_C(y - \gamma_n \nabla f^{S\tau_n} y)\|^2 &= (1 - \gamma_n \tau_n)^2 \|x - y\|^2 \\ &\leq \|x - y\|^2. \qquad (12) \end{aligned}$$

Lemma 3.2

The sequence $\{x_n\}$ generated by (11) is bounded.

Sketch of the proof

Let $p \in \Gamma$.

- $$\|u_n - p\| \leq \|x_n - p\| + \gamma_n\tau_n\|p\|. \quad (13)$$

Using Lemma 5 and (13), we get

- $$\|x_{n+1} - p\| \leq \|x_n - p\| + \gamma_n\tau_n\|p\|. \quad (14)$$

The result follows by applying Lemma 4 to (14).

Main Result

Theorem 3.3

The sequence $\{x_n\}$ generated by Algorithm (11) converges weakly to a point in Γ.

Sketch of the proof

Let $t_n := x_n - \gamma_n \nabla f^{S\tau_n}(x_n)$ and $x \in \Gamma$.

$$\begin{aligned} \|x_{n+1} - x\|^2 \leq\ & \|x_n - x\|^2 - \gamma_n(1 - \gamma_n\|A\|^2)\|Ax_n - SP_Q Ax_n\|^2 \\ & -\gamma_n\tau_n\|x_n\|^2(2 - \gamma_n\tau_n) \\ & -\gamma_n\|P_Q Ax_n - Ax_n\|^2 + 2\gamma_n\tau_n\langle x + \gamma_n\nabla f^S x_n, x_n\rangle \\ & -\|t_n - P_C t_n\|^2 - \alpha_n(\beta_n - \alpha_n)(1 - \beta_n L)^2\|Tu_n - u_n\|^2 \end{aligned} \tag{15}$$

Main Result

Proof (contd)

Rearranging and taking the limit of (15) as $n \to \infty$, we see that

$$\|Ax_n - SP_Q Ax_n\| \to 0 \text{ as } n \to \infty, \tag{16}$$

$$\|P_Q Ax_n - Ax_n\| \to 0 \text{ as } n \to \infty, \tag{17}$$

and

$$\|t_n - P_C t_n\| \to 0, \|Tu_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{18}$$

It follows from (16) that

$$\begin{aligned} \|t_n - x_n\| &= \|x_n - \gamma_n \nabla f^{S\tau_n}(x_n) - x_n\| \\ &\leq \gamma_n \|A^*\| \|Ax_n - SP_Q Ax_n\| + \gamma_n \tau_n \|x_n\| \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{19}$$

Main Result

Proof (contd)

Using (18) and the fact that T is L-Lipschitzian, we find that

$$\begin{aligned} \|x_{n+1} - u_n\| &= \alpha_n \|T((1-\beta_n)u_n + \beta_n Tu_n) - u_n\| \\ &\leq \alpha_n(L\beta_n + 1)\|Tu_n - u_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{20}$$

Furthermore, from (16) and (17) it follows that

$$\|P_Q Ax_n - SP_Q Ax_n\| \to 0 \text{ as } n \to \infty. \tag{21}$$

Choose a weakly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and let p be its weak limit. Then we have $u_{n_i} \rightharpoonup p$ and $Ax_{n_i} \rightharpoonup Ap$ as $i \to \infty$. Since $I - T$ and $I - S$ are demiclosed at zero by Lemma 2, it follows that $p \in Fix(T)$ and $Ap \in Fix(S)$. Furthermore, since C and Q are weakly closed, it also follows that $p \in C$ and $Ap \in Q$. Thus $p \in \Gamma$. The conclusion thus follows by invoking Lemma 1, Lemma 3.2 and Lemma 6.

Application

Split feasibility and convex minimization problem (SFCMP)

Let $g : H_1 \to \mathbb{R}$ be a convex and differentiable function with L-Lipschitz continuous gradient ∇g and let $A : H_1 \to H_2$ be a bounded linear operator. Consider the SFCMP: Find $x^* \in H_1$ such that

$$x^* \in C \cap \arg\min_{x \in H_1} g(x) \text{ and } Ax^* \in Q. \tag{22}$$

The SFCMP serves as a model for some applied problems in image processing and signal recovery such as finding the minimum energy for bandlimited signals, and constrained denoising problems.
The mapping $\nabla g : H_1 \to H_1$ satisfies the following inequality:

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq \frac{1}{L} \|\nabla g(x) - \nabla g(y)\|^2. \tag{23}$$

From (23) ∇g is monotone. Consequently, $I - \nabla g$ is a Lipschitzian pseudocontractive mapping with Lipschitz constant $(1 + L)$.

Application

We denote the solution set of the SFCMP (22) by Γ_4. The SFCMP (22) can be recast as follows: Find

$$x^* \in H_1 \text{ such that } x^* \in C \cap \text{zer}\,(\nabla g) \text{ and } Ax^* \in Q.$$

We therefore obtain the following theorem regarding the approximation of solutions to the SFCMP (22).

Theorem 4.1

Let H_1, H_2 be real Hilbert spaces, and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of A. Let $g : H_1 \to \mathbb{R}$ be a convex and differentiable function with L- Lipschitz continuous gradient ∇g. Suppose that $\Gamma_4 \neq \emptyset$ and let $\{x_n\}$ be the sequence generated by (11) where $T = I - \nabla g$, $S = I$, and the conditions (i), (ii) and (iii*) $0 < c < \alpha_n < d < \beta_n < e < \frac{1}{1+\sqrt{1+(1+L)^2}}$. Then the sequence $\{x_n\}$ converges weakly to a point in Γ_4.

Numerical Example

Let $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R} \times \mathbb{R}$ equipped with the Euclidean inner products and the induced norms denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Let $C = [0, \infty)$ and $Q = \{x \in H_2 : 1 \leq \|x\| \leq 3\}$. Define $T : H_1 \to H_1$ by

$$Tx := \begin{cases} x - 1 + \frac{4}{x+1} & \text{if } x \in [0, \infty), \\ 3 & \text{otherwise}, \end{cases}$$

and $S : H_2 \to H_2$ by

$$Sx := (\frac{-x_1}{2} + \frac{3}{2}, \frac{x_2}{3} + \frac{1}{3}) \quad \forall x = (x_1, x_2) \in H_2.$$

T is Lipschitzian Pseudocontractive for $x, y \in [0, \infty)$ (Yao et al. 2014). If $x \in [0, \infty)$ and $y \in (-\infty, 0)$, then $(x - y) > 0$ and $y - \frac{4x}{x+1} < 0$. Here,

$$\langle Tx - Ty, x - y \rangle \leq |x - y|^2$$

$$|Tx - Ty| \leq 5|x - y|.$$

The above inequalities obviously hold if $x, y \in (-\infty, 0)$. Therefore, it follows that T is an L-Lipschitzian pseudocontractive mapping with $L = 5$.

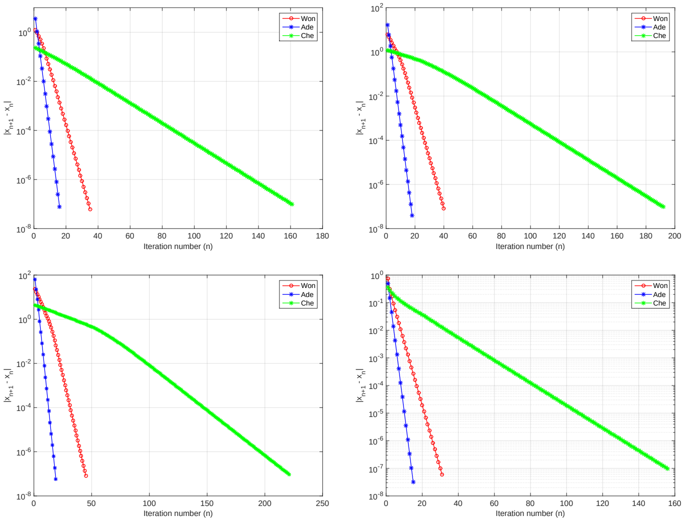
Numerical Example

S is nonexpansive. Let $A : H_1 \to H_2$ be defined by $Ax = (\frac{x}{3}, \frac{x}{6}) \quad \forall\, x \in H_1$. In this example $\Gamma = \{3\}$. We choose $\tau_n = \frac{1}{(n^2+1)}$, $\gamma_n = \frac{2n}{n+5}$, $\delta_n = \frac{3n}{100n+1}$, $\beta_n = \frac{7n}{100n+1}$, $\alpha_n = \frac{13n}{100n+1}$ for all $n \geq 1$. We compare the performance of our Algorithm 15 (Ade) with that of algorithms (9) (Won) and (8) (Che) using different initial values x_1. SC:error $= |x_{n+1} - x_n| < 10^{-7}$.

Table: Numerical results.

		Ade	Won	Che
Case Ia	CPU time (sec)	0.0011	0.0233	0.0044
$x_1 = 6$	No of Iter.	16	35	161
Case Ib	CPU time (sec)	0.0012	0.0023	0.0030
$x_1 = 25.65$	No. of Iter.	18	40	192
Case Ic	CPU time (sec)	0.0022	0.0238	0.0028
$x_1 = 98.22$	No of Iter.	19	45	221
Case Id	CPU time (sec)	0.0015	0.0025	0.0086
$x_1 = 0.222$	No of Iter.	15	31	156

Numerical Example



Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id

Conclusion

Conclusion

We have studied the split feasibility and fixed point problem for Lipschitzian pseudocontractive and nonexpansive mappings in real Hilbert spaces. By combining the gradient-projection method with Ishikawa iterations, we have proposed a new iterative scheme that involves the computation of just two metric projections per iteration. We have established a weak convergence theorem and have given an application of our main result.

Thank You

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