Asymptotic Behavior of Some Expansive Type Difference Equations and Evolution Systems

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Abstract

In this talk, after reviewing some old results in nonlinear ergodic theory and their applications to the study of the asymptotic behavior of quasi-autonomous dissipative systems, we concentrate on first order expansive type evolution and difference equations and present some old and new results on the asymptotic behavior of the solutions, as well as periodic solutions to such systems.

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- *H* is a real Hilbert space endowed with scalar product $\langle \cdot, \cdot \rangle$, induced norm $\| \cdot \|$ and identity operator *I*.

- $A : D(A) \subset H \to H$ is a (possibly multivalued) maximal monotone operator.

- By $\to$ and $\rightharpoonup$ we respectively denote strong and weak convergence in *H*.

For a curve $u : [0, +\infty) \to H$, we denote

- $F(u(t)) = \{q \in H : \lim_{t \to +\infty} \|u(t) - q\| \text{ exists}\}$,

- $\sigma_T = \frac{1}{T} \int_0^T u(t) dt$,

- $\omega_w(u(t))$ the set of all weak cluster points of the net $u(t)$.

Definition

A mapping $T : D \subset H \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \forall x, y \in D,$$

where D is a nonempty subset of H.

The following theorem was proved in 1965, independently by Browder, Kirk and Gohde.

Theorem

Suppose that $T : D \to D$ is a nonexpansive mapping, where D is a nonempty, bounded, closed and convex subset of H, then T has a fixed point and $\mathrm{Fix}(T) := \{x : Tx = x\}$ is closed and convex.

Although in the Banach fixed point theorem, all orbits converge to the unique fixed point of *T*, this fact does not hold for a nonexpansive mapping, and orbits may not converge at all. Baillon, in 1975, proved that the Cesaro means of the Picard iterates of any nonexpansive mapping T, always converge weakly to a fixed point of *T*, provided that Fix$T \neq \emptyset$.

Theorem

Let C be a nonempty, closed and convex subset of H, and T be a nonexpansive mapping from C into itself. If the set Fix(*T*) *is nonempty, then for each x ∈ C, the Cesaro means*

$$S_n(x) = \frac{1}{n}\sum_{k=1}^{n-1} T^k x$$

converges weakly to some y ∈ Fix T.

If C is not convex, then Fix(T) may be empty, and then Baillon's proof is not applicable. To avoid the convexity assumption on C, we are going to introduce the notion of nonexpansive curves.

Definition

- The curve $u(t)$ in H is nonexpansive if for all $r, s, h \geq 0$, we have $\|u(r + h) - u(s + h)\| \leq \|u(r) - u(s)\|$.
- $u(t)$ is an almost nonexpansive curve if for all $r, s, h \geq 0$, we have $\|u_{r+h} - u_{s+h}\|^2 \leq \|u(r) - u(s)\|^2 + \varepsilon(r, s)$, where $\lim_{r,s \to +\infty} \varepsilon(r, s) = 0$.

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Preliminaries

Definition (Asymptotic center)

Given a bounded curve $u(t)$ in H, the asymptotic center c of $u(t)$ is defined as follows: for every $q \epsilon H$, let
$\phi(q) = \limsup_{t \to +\infty} \|u(t) - q\|^2$. Then ϕ is a continuous and strictly convex function on H, satisfying $\phi(q) \to +\infty$ as $\|q\| \to +\infty$. Therefore ϕ achieves its minimum on H at a unique point c, called the asymptotic center of the net $u(t)$.

Theorem (BDR, 1990)

Let $u(t)$ be an almost nonexpansive curve in H. Then the following are equivalent:

(i) $F(u(t)) \neq \emptyset$

(ii) $\liminf_{T \to +\infty} \|\sigma_T\| < +\infty$.

(iii) σ_T *converges weakly to* $p \in H$.

Moreover under these conditions we have:

- $\overline{\mathrm{conv}}\, (\omega_w(u(t))) \cap F(u(t)) = \{p\}$.
- p *is the asymptotic center of the net* $u(t)$.

Definition

- The curve $u(t)$ in H is asymptotically regular if for all $h > 0$, $u(t + h) - u(t) \rightarrow 0$ as $t \rightarrow +\infty$.

- $u(t)$ is a weakly asymptotically regular curve in H if $u(t + h) - u(t) \rightharpoonup 0$ as $t \rightarrow +\infty$.

Theorem (BDR, 1990)

Let u(t) be a weakly asymptotically regular and almost nonexpansive curve in H. Then the following are equivalent:

(i) $F(u(t)) \neq \emptyset$

(ii) $\liminf_{t \to +\infty} \|u(t)\| < +\infty$.

(iii) u(t) converges weakly to $p \in H$.

Backgrounds

Asymptotic behavior of dissipative systems of the form

$$-\dot{u}(t) \in Au(t),$$
$$u(0) = u_0, \tag{1}$$

where A is a maximal monotone operator in H and $u_0 \in \overline{D(A)}$ is arbitrary have been studied by several authors in the 1970s.

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Quasi-autonomous Dissipative Systems

Backgrounds

Using the notion of nonexpansive and almost nonexpansive curves in H, Djafari-Rouhani extended their result to the case of quasi-autonomous

$$-\dot{u}(t) \in Au(t) + f(t),$$
$$u(0) = u_0,$$

(2)

without assuming A to have a nonempty zero set.

Theorem (BDR, 1990)

If u is a weak solution of the system (2) on every interval $[0, T]$, and satisfies $\sup_{t>0} \|u(t)\| < +\infty$, and if $f - f_\infty \in L^1((0, +\infty); H)$ for some $f_\infty \in H$, then $\sigma_T = (I/T)\int_0^T u(t)dt$ converges weakly to the asymptotic center of the curve u(t).

Theorem (BDR, 1990)

If u is a weak solution of the system (2) on every interval $[0, T]$, and satisfies $\sup_{t>0} \|u(t)\| < +\infty$ and for all $h \geq 0$, $u(t + h) - u(t) \to 0$ as $t \to +\infty$, and if $f - f_\infty \in L^1((0, +\infty); H)$ for some $f_\infty \in H$, then u(t) converges weakly as $t \to +\infty$ to the asymptotic center of the curve u(t).

Theorem (Brezis-Browder, 1977)(Extension by BDR, 1990)

If u is a weak solution of the system ([2](#)) on every interval $[0, T]$, and satisfies $\lim\langle u(t), u(t + h)\rangle = \alpha(h)$ exists uniformly in $h \geq 0$, then $\sigma_T = (1/T) \int_0^T u(t)dt$ converges strongly as $T \to +\infty$, to the asymptotic center of the curve $u(t)$.

Backgrounds

Definition

A map $T : D(T) \subset H \to H$ is said to be expansive if $\|x - y\| \leq \|Tx - Ty\|$, for all $x, y \in D(T)$.

The study of expansive maps began with L. Nirenberg's problem concerning the surjectivity of such self maps of H. Djafari Rouhani proved the first mean ergodic theorem for expansive mappings, where applications to autonomous evolution systems of expansive type in H were also considered.

Definition

For a sequence $u_n \in H$, we define

$$E_1 = \{q \in H : \text{the sequence } \|u_n - q\| \text{ is nondecreasing}\}.$$

Theorem (BDR, 2001)

Let D be a nonempty subset of H, T an expansive self-mapping of D and $u_0 \in D$. Let $u_n = T^n u_0$ and $s_n = \dfrac{1}{n}\sum\limits_{k=1}^{n-1} u_k$

(i) *If $\liminf_{n\to+\infty} \|s_n\| < +\infty$ and $\|u_n\| = o(\sqrt{n})$, then the weak limit q of any weakly convergent subsequence s_{n_i} of s_n belongs to E_1.*

(ii) *If in addition to (i), $\liminf_{n\to+\infty} \|u_n\| < +\infty$, then u_n is bounded and s_n converges weakly to the asymptotic center p of u_n. Moreover we have $p = \lim_{n\to+\infty} P_{E_1} u_n$.*

(iii) *If in addition to (ii), u_n is weakly asymptotically regular, then u_n converges weakly to p as $n \to +\infty$.*

(iv) *If $\lim_{n\to+\infty} \|u_n\|$ exists, then s_n converges strongly to the asymptotic center p of u_n, and moreover in addition to $p = \lim_{n\to+\infty} P_{E_1} u_n$, we have $p = P_K 0$, where $K_n = clco(u_n)$ and $K = \cap_{n=0}^{\infty} K_n$.*

Definition

An expansive curve u in H is a curve satisfying
$\|u(t + h) - u(s + h)\| \geq \|u(t) - u(s)\|$ for all $s, t, h \geq 0$.

Theorem (BDR, 2001)

Let u be an expansive curve in H and $\sigma_T = \frac{1}{T}\int^T u(t)dt$ for T > 0.

(i) *If $\liminf_{T\to+\infty} \|\sigma_T\| < +\infty$ and $\|u(t)\| = o(\sqrt{t})$,then the weak limit q of any weakly convergent subsequence of σ_T of s_n belongs to E_1.*

(ii) *If in addition to (i), $\liminf_{t\to+\infty} \|u(t)\| < +\infty$, then u is a bounded curve and σ_T converges weakly to the asymptotic center p of u(t). Moreover we have $p = \lim_{t\to+\infty} P_{E_1} u(t)$.*

(iii) *If in addition to (ii), u is weakly asymptotically regular, then u(t) converges weakly to p as $t \to +\infty$.*

(iv) *If $\lim_{t\to+\infty} \|u(t)\|$ exists, then σ_T converges strongly to the asymptotic center p of u(t), and moreover in addition to $p = \lim_{t\to+\infty} P_{E_1} u(t)$, we have $p = P_K 0$, where $K_t = \text{clco}\{u(s); s \geq t\}$ and $K = \cap_{t\geq 0} K_t$.*

Lemma (BDR, 2001)

Let A be a monotone operator in H; if u is weak solution of

$$\dot{u}(t) \in Au(t),$$
$$u(0) = u_0,$$

(3)

on [0, T] for every T > 0, then u is an expansive curve in H.

Theorem

With the same assumptions as in the above lemma, the statements (i), (ii), (iii) and (iv) in the previous theorem describe the asymptotic behavior of a solution u(t) of system (3).

Remark (Comparing these results to the corresponding ones for dissipative systems)

If A is maximal monotone and $u_0 \in \overline{D(A)}$, then the system

$$-\dot{u}(t) \in Au(t),$$
$$u(0) = u_0, \tag{4}$$

has a unique weak solution and $p \in A^{-1}(0)$, whereas for solutions to

$$\dot{u}(t) \in Au(t),$$
$$u(0) = u_0, \tag{5}$$

neither existence nor uniqueness is not guaranteed in this case.

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Almost Expansive Curves

Definition

The curve u in H is called almost expansive if
$$\limsup_{s,t\to+\infty} \; [\sup_{h\geq 0}(\|u(s) - u(t)\|^2 \;\; -\|u(s+h) - u(t+h)\|^2) \;] \leq 0,$$

i.e. for every $\varepsilon > 0$, there exists $t_0 \geq 0$ such that for all $s, t \geq t_0$, and for all $h \geq 0$, we have

$$\|u(s) - u(t)\|^2 \leq \|u(s+h) - u(t+h)\|^2 \;\; + \varepsilon.$$

Remark

We note that if u is bounded, then this definition is equivalent to

$$\limsup_{s,t\to+\infty} \sup_{h\geq 0}(\|u(s) - u(t)\| - \|u(s+h) - u(t+h)\|) \leq 0.$$

Ergodic Theorems for almost expansive curves in H

Proposition (BDR, 2004)

If $\liminf_{T\to+\infty} \|\sigma_T\| < +\infty$ and $\|u(t)\| = o(\sqrt{t})$, then either the weak limit q of any weakly convergent subsequence σ_{T_n} of σ_T belongs to $F(u)$ or $\|u(t)\| \to +\infty$ as $t \to +\infty$.

Theorem (BDR, 2004)

Let u be an almost expansive curve in H. Assume $\liminf_{T\to+\infty} \|\sigma_T\| < +\infty$, $\liminf_{t\to+\infty} \|u(t)\| < +\infty$ and $\|u(t)\| = o(\sqrt{t})$. Then u is bounded and σ_T converges weakly as $T \to +\infty$, to the asymptotic center p of u.

Theorem (BDR, 2004)

Let u be an almost expansive curve in H such that
$\liminf_{T\to+\infty}\|\sigma_T\| < +\infty$, $\liminf_{t\to+\infty}\|u(t)\| < +\infty$ *and*
$\|u(t)\| = o(\sqrt{t})$.*Then u(t) converges weakly as* $t \to +\infty$ *to the asymptotic center p of u, if and only if u is weakly asymptotically regular.*

Strong ergodic theorem for almost expansive curves in H

Theorem (BDR, 2004)

Let u be an almost expansive curve in H. If $0 \in F(u)$, then σ_T converges strongly as $T \to +\infty$ to the asymptotic center p of u. Moreover we have $p = P_K 0$, where $K_t = \text{clco}\{u(s); s \geq t\}$ and $K = \cap_{t \geq 0} K_t$.

Strong convergence theorem for almost expansive curves in H

Theorem (BDR, 2004)

Let u be an almost expansive curve in H. Assume u is asymptotically regular. Then $\lim_{t \to +\infty} u(t) = p = P_K 0$ where p is the asymptotic center of u and $K_t = \text{clco}\{u(s); s \geq t\}$ and $K = \bigcap_{t \geq 0} K_t$.

Proposition (BDR, 2004)

If u is a weak solution of

$$\dot{u}(t) + f(t) \in Au(t),$$
$$u(0) = u_0, \qquad\qquad (6)$$

on [0, T] for every T > 0, and if $\sup_{t \geq 0} \|u(t)\| < +\infty$ and

$$\lim_{s,r \to +\infty} \int_s^{+\infty} \|f(\theta + (r-s)) - f(\theta)\| d\theta = 0, \qquad (*)$$

then the curve u is almost expansive in H.

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Quasi-autonomous Expansive Type Evolution

Asymptotic Behavior of Quasi-Autonomous Expansive Type Evolution Systems

Theorem (BDR, 2004)

Assume u is a weak solution of ([6](#)) on every interval $[0, T]$ and $\sup_{t \geq 0} \|u(t)\| < +\infty$. Assume $f - f_\infty \in L^1((0, +\infty); H)$ for some $f_\infty \in H$. Then the following hold:

  (i) $\sigma_T \to p$ as $T \to +\infty$, where p is the asymptotic center of u.

  (ii) $u(t) \to p$ as $t \to +\infty$, if and only if u is weakly asymptotically regular.

 (iii) *If $\lim_{t \to +\infty} \|u(t)\|$ exists, then $\lim_{T \to +\infty} \sigma_T = p = P_K 0$, where K is already defined.*

 (iv) $\lim_{t \to +\infty} u(t) = p = P_K 0$ *if and only if u is asymptotically regular.*

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Unlike the dissipative case, the systems of the form

$$\dot{u}(t) \in Au(t)$$
$$u(0) \in D(A), \tag{7}$$

are "strongly ill-posed" as shown for example by considering the simple linear case of $A = -\Delta$ with Dirichlet boundary conditions, where we obtain the heat equation with a final Cauchy data which is not generally solvable. A similar situation occurs for the backward discretization

$$u_{n+1} - u_n \in \lambda_n A u_{n+1}. \tag{8}$$

Hence, we consider the following forward discretization:

$$u_{n+1} - u_n \in \lambda_n A u_n, \tag{9}$$

which is well-posed, and the sequence u_n is always well-defined.

Similar to the continuous case, by introducing the notion of almost expansive sequences and study their asymptotic behavior under some suitable conditions, we describe the asymptotic behavior of the solution to (9).

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Expansive Type Difference Equation

Definition

A sequence u_n in H is said to be almost expansive if for all $i, j, k \geq 0$, we have

$$\limsup_{i,j \to \infty}[\sup_{k \geq 0}(\| u_i - u_j \|^2 - \| u_{i+k} - u_{j+k} \|^2)] \leq 0.$$

i.e $\forall \varepsilon > 0$, $\exists N_0$ such that $\forall i, j \geq N_0$, $\forall k \geq 0$,

$\|u_i - u_j\|^2 \leq \|u_{i+k} - u_{j+k}\|^2 + \varepsilon$.

We note that if u_n is bounded, then this definition is equivalent to

$$\limsup_{i,j \to \infty}[\sup_{k \geq 0}(\| u_i - u_j \| - \| u_{i+k} - u_{j+k} \|)] \leq 0.$$

Proposition (BDR-MRP, 2020)

Let λ_n be a nondecreasing sequence of positive numbers such that

$$\limsup_{\substack{j \geq i \\ i, j \to +\infty}} \sum_{l=i}^{+\infty} \left(\frac{\lambda_{(j-i)+l}}{\lambda_l} - 1 \right) = 0 \qquad (*)$$

If u_n is a bounded solution to ([9]), then u_n is almost expansive.

Remark

Condition $(*)$ in the above proposition is in particular satisfied if $\sup_{n \geq 1} \lambda_n \leq \lambda$ for some $\lambda > 0$, and $\frac{\lambda}{a_n+1} \leq \lambda_n$ for some $a_n \in l^1$.

Example

The sequence $\lambda_n = \frac{n^2}{1+n^2}$ satisfies the conditions of the above proposition.

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Expansive Type Difference Equation

Theorem (BDR-MRP, 2020)

Assume that λ_n is a nondecreasing sequence satisfying condition $(*)$, and u_n is a bounded solution to ([9]). Then the following hold:

(i) $s_n \rightharpoonup p$, as $n \to +\infty$ where p is the asymptotic center of u_n.

(ii) $u_n \rightharpoonup p$, as $n \to +\infty$ if and only if u is weakly asymptotically regular.

(iii) If $\lim_{n\to+\infty} \|u_n\|$ exists, then $\lim_{n\to+\infty} s_n = p = P_K 0$, where K is as already defined.

(iv) $\lim_{n\to+\infty} u_n = p = P_K 0$ if and only if u_n is asymptotically regular.

If A has a nonempty zero set, then we can conclude stronger results:

Theorem (BDR-MRP, 2020)

Let u_n be the sequence generated by (9), where $A^{-1}(0) \neq \emptyset$ and $\liminf_{n \to +\infty} \lambda_n \geq \lambda$ for some $\lambda > 0$. If u_n is bounded, then there exists some $p \in A^{-1}(0)$ such that $u_n \to p$ as $n \to +\infty$. Otherwise $\|u_n\| \to +\infty$ as $n \to +\infty$.

Remark

If the step size λ_n goes to infinity as $n \to +\infty$, then the existence of a bounded solution to (9) implies that $A^{-1}(0) \neq \emptyset$. In fact, let u_n be a bounded solution to (9) and $b_n = \dfrac{u_{n+1} - u_n}{\lambda_n}$. Clearly $b_n \in Au_n$ and $b_n \to 0$.

Since u_n is bounded, there exist some $q \in H$ and a subsequence u_{n_k} such that $u_{n_k} \to q$ as $k \to +\infty$. Now the maximality of A implies that $q \in A^{-1}(0)$.

Definition

The operator $T : D(T) \subset H \to H$ is said to be α-expansive if

$$\alpha\|x - y\| \leq \|Tx - Ty\|, \qquad \forall x, y \in D(T).$$

If $\alpha = 1$, we say that T is expansive.

Lemma

If $T : H \to H$ is α-expansive and onto, then T^{-1} exists and it is $\frac{1}{\alpha}$-Lipschitz.

Theorem (BDR-MRP, 2020)

Suppose that A is a single-valued and maximal strongly monotone operator in H. If λ_n is a periodic sequence with period N, then there exists an N-periodic solution to (9).

Remark

The following simple example shows that the above theorem does not hold for a general maximal monotone operator A, not even for subdifferentials of proper, convex and lower semicontinuous functions, or for inverse strongly monotone operators. Let $A : R \rightarrow R$ be the constant function $A \equiv 1$, and $\lambda_n \equiv 1$. Then (9) reduces to $u_{n+1} = u_n + 1$, which shows that the sequence u_n tends to $+\infty$, as $n \rightarrow +\infty$, for all $u_0 \in R$. Therefore it does not have a periodic solution.

Theorem (BDR-MRP, 2020)

Assume that A is a single-valued and maximal monotone operator in H, and the sequence λ_n is periodic with period N. If (9) has an N-periodic solution w_n, then every bounded solution to (9) is also periodic with period N, and differs from w_n by an additive constant.

Remark

We show by an example that the existence of periodic solutions does not imply the boundedness of all solutions to (9). Let $D = [0, 1]$, $A = (I - P_D)$, and $\lambda_n \equiv 1$. Then (9) reduces to $u_{n+1} = 2u_n - P_D u_n$. If we choose $u_0 = 0$, then $u_n \equiv 0$, which is a periodic solution with period N for all $N \in$ N. But if we choose $u_0 = 2$, then $u_{n+1} = 2u_n - 1$, which clearly goes to $+\infty$, as $n \to +\infty$.

Remark

Now we compare the results of this section, to the results in
[Rouhani-Khatibzadeh,2012] for the first order difference
equation

$$u_{n-1} - u_n \in \lambda_n A u_n + f_n,$$
$$u_0 = x. \tag{10}$$

In [Rouhani-Khatibzadeh,2012], the authors proved the existence
of an N-periodic solution w_n to (10), where λ_n and f_n are periodic
with period N, and (10) has a solution u_n satisfying
$\liminf_{n\to+\infty} \|s_n\| < +\infty$. They also proved that $u_n - w_n \to 0$, as
$n \to +\infty$, and any two periodic solutions differ by an additive
constant.

An important example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function. Inspired by its applications in economics, the study of functions that are not convex but have convex sublevel sets have received a particular attention. The functions with convex sublevel sets are called *quasiconvex*.

Definition

- A function $\phi : H \rightarrow (-\infty, +\infty]$ is called quasiconvex if

    $\phi(\lambda x + (1-\lambda)y) \leq \max\{\phi(x), \phi(y)\}$, $\forall x, y \in H$ and $\forall \lambda \in [0,1]$.

- ϕ is *strongly quasiconvex* if there exists $\alpha > 0$ such that

    $\phi(\lambda x + (1-\lambda)y) \leq \max\{\phi(x), \phi(y)\} - \alpha\lambda(1-\lambda)\|x-y\|^2$,
    $\forall x, y \in H$ and $\forall \lambda \in [0,1]$.

There are many attempts to generalize the notion of subdifferential for non convex functions. However in any circumstance, the subdifferential of a quasiconvex function is not monotone. On the other hand, if $\phi: H \to$ R is Gâteaux differentiable, then the following characterization for a quasiconvex function ϕ holds:

ϕ is quasiconvex on $H \iff (\forall x, y \in H, \phi(y) \leq \phi(x) \Rightarrow \langle \nabla\phi(x), x-y \rangle \geq 0)$

The above characterization may prove to be useful given the lack of monotonicity.

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A Gradient System of Expansive Type

We consider the following differential equation

$$\dot{u}(t) = \nabla\phi(u(t)) + f(t), \qquad t \in [0, +\infty),$$
$$u(0) = u_0 \in H, \qquad\qquad\qquad (11)$$

where $\phi: H \to \mathbb{R}$ is a differentiable quasiconvex function such that $\nabla\phi$ is Lipschitz continuous and $f \in W^{1,1}((0, +\infty); H)$.
The Lipschitz continuity of $\nabla\phi$ implies that the system (11) with an initial condition has a unique solution $u(t)$. In order to study the asymptotic behavior of such a solution, we define

$$L(u) = \{y \in H : \exists T > 0 \text{ s.t. } \phi(y) \le \phi(u(t)) \quad \forall t \ge T\}.$$

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The set of all global minimizers of ϕ is denoted by argmin ϕ. Clearly, argmin $\phi \subset L(u)$.

Proposition (BDR-MRP, 2021)

Assume that $u(t)$ is a solution to (11). For an arbitrary interval $[a, b]$, where $b \geq a \geq 0$, and each $y \in L(u)$, we have

$$\|u(a) - y\| \leq \|u(b) - y\| + \int_a^b \|f(t)\| dt, \qquad (12)$$

and therefore $\lim_{t \to +\infty} \|u(t) - y\|$ exists (it may be infinite).

Proposition (BDR-MRP, 2021)

Let $u(t)$ be a solution to (11) such that $\liminf_{t \to +\infty} \|u(t)\| < +\infty$. Then $\lim_{t \to +\infty} \nabla \phi(u(t)) = 0$ and $\lim_{t \to +\infty} \phi(u(t))$ exists and is finite.

Proposition (BDR-MRP, 2021)

If $u(t)$ is a solution to (11) such that $\lim \inf_{t\to+\infty} \|u(t)\| < +\infty$, *then $L(u) \neq \emptyset$ and u is bounded.*

Theorem (BDR-MRP, 2021)

Let $u(t)$ be a solution to (11). If $\lim \inf_{t\to+\infty} \|u(t)\| < +\infty$, *then there exists some $p \in (\nabla\phi)^{-1}(0)$ such that $u(t) \to p$ as $t \to +\infty$, and if $p \notin \operatorname{argmin} \phi$, the convergence is strong. If $u(t)$ is unbounded, then $\|u(t)\| \to +\infty$ as $t \to +\infty$.*

Remark

The above theorem shows that if $(\nabla\phi)^{-1}(0) = \emptyset$, then for any solution to (11), we have $\lim_{t\to+\infty} \|u(t)\| = +\infty$.

Theorem (BDR-MRP, 2021)

If either one of the following assumptions is satisfied, then bounded solutions to ([11](#)) converge strongly to some point in $(\nabla\phi)^{-1}(0)$:

(i) *Sublevel sets of ϕ are compact.*

(ii) int $L(u) \neq \varnothing$

Theorem (BDR-MRP, 2021)

Assume that $\phi: H \to \mathbb{R}$ is a strongly quasiconvex function and $u(t)$ is a bounded solution to ([11](#)). Then argmin *ϕ is a singleton and $u(t)$ converges strongly to the unique minimizer of ϕ.*

We consider the following discrete version of (11):

$$u_{n+1} - u_n = \lambda_n \nabla\phi(u_n) + f_n,$$
$$u_0 = x \in H,$$

(13)

where $f_n \in l^1$, $\lambda_n \geq \varepsilon$ for some $\varepsilon > 0$, and $\phi: H \to$ R is a differentiable quasiconvex function such that $\nabla\phi$ is Lipschitz continuous with Lipschitz constant K.
In order to study the asymptotic behavior of u_n, we define the following discrete version of $L(u)$:

$$L(u_n) = \{y \in H : \exists N > 0 \text{ s.t. } \phi(y) \leq \phi(u_n) \quad \forall n \geq N\}.$$

Proposition (BDR-MRP, 2021)

Let u_n be the sequence generated by (13). For each $y \in L(u_n)$, and $k < m$, we have

$$\|u_k - y\| \leq \|u_m - y\| + \sum_{n=k}^{m-1} \|f_n\|, \tag{14}$$

and consequently $\lim_{n \to +\infty} \|u_n - y\|$ exists (it may be infinite).

Proposition (BDR-MRP, 2021)

Let u_n be a solution to (13) such that $\liminf_{n \to +\infty} \|u_n\| < +\infty$. Then $L(u_n)$ is nonempty if and only if $\lim_{n \to +\infty} \phi(u_n)$ exists, and in this case u_n is bounded.

Remark

If ϕ is convex, we can omit the condition that $\nabla \phi$ is Lipschitz continuous.

Proposition (BDR-MRP, 2021)

Assume that u_n is a solution to (13) such that $\liminf_{n \to +\infty} \|u_n\| < +\infty$. If either one of the following conditions is satisfied, then $L(u_n)$ is nonempty.

(i) *ϕ is convex and the sequence of step sizes λ_n is bounded above.*

(ii) *$\limsup_{n \to +\infty} \lambda_n < \frac{2}{K}$.*

Open problem. In the continuous case, we showed that if $\liminf_{t \to +\infty} \|u(t)\| < +\infty$, then $L(u) \neq \emptyset$ However, in the discrete case, we do not know whether without any additional assumption, $\liminf_{n \to +\infty} \|u_n\| < +\infty$ implies that $L(u_n)$ is nonempty.

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Dissipative Systems
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Expansive-type Systems
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A Gradient System of Expansive Type

Theorem (BDR-MRP, 2021)

Let u_n be the sequence generated by (13) and $L(u_n) \neq \emptyset$. If $\liminf_{n \to +\infty} \|u_n\| < +\infty$, then there exists some $p \in (\nabla\phi)^{-1}(0)$ such that $u_n \to p$ as $n \to +\infty$ and if $p \notin \text{argmin } \phi$ the convergence is strong. If u_n is unbounded, then $\|u_n\| \to +\infty$, as $n \to +\infty$.

Example

Assume that $\phi: R \to R$ is the function defined as $\varphi(x) = \arctan(x^3)$ and consider (13) with $\lambda_n = (2/3) n$ and $f_n \equiv 0$. Then all the assumptions of the above theorem are satisfied. The following table compares 1000 iterations of the sequence u_n given by (13) with two different initial values $u_0 = -0.5$ and $u_0 = 1$. The numerical results show that for $u_0 = -0.5$, $u_n \to 0 \in (\nabla\phi)^{-1}(0)$ and for $u_0 = 1$, u_n slowly goes to infinity.

Numerical results

n	u_n	u_n
0	-0.5	1
1	-0.00769231	2
10	-0.00404869	3.63765
20	-0.00171074	4.68854
30	-0.0008858	5.46951
40	-0.000533135	6.11128
50	-0.000354164	6.66517
60	-0.000251763	7.15741
70	-0.000187942	7.60348
80	-0.000145564	8.01339
90	-0.000116023	8.39404
100	-0.0000946225	8.7504
1000	-9.94968×10^{-7}	21.8786

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Dissipative Systems
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Expansive type
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A Gradient System of Expansive Type

Theorem (BDR-MRP, 2021)

Assume that u_n is a bounded sequence which satisfies (13)
*and $L(u_n) \neq \emptyset$ If either one of the following assumptions is
satisfied, then u_n converges strongly to some point in
$(\nabla \phi)^{-1}(0)$:*

  (i) Sublevel sets of ϕ are compact.

  (ii) int $L(u_n) \neq \emptyset$

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Thank You