A survey of selected results in fixed point theory

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THEOREM 1 (F. Cammaroto and A. Chinnì , 1996) . - *Let E be a Banach space, $f : E \to E$ and $\varphi : E \to [0, +\infty[$ a lower semicontinuous function such that*

$$\|x - f(x)\| \leq \varphi(x) - \varphi(f(x))$$

for all $x \in X$. Moreover, assume that

$$\limsup_{\|x\| \to +\infty} \frac{\varphi(x)}{\|x\|} < 1 \ .$$

Then, the convex hull of $\mathrm{Fix}(f)$ is dense in X.

Let $(H, \langle \cdot, \cdot \rangle)$ will be a real Hilbert space.

For each $r > 0$, we put

$$B_r = \{x \in H : \|x\|^2 < r\}$$

and

$$S_r = \{x \in H : \|x\|^2 = r\} .$$

A continuous operator $T : H \to H$ is said to be a potential operator if there exists a Gâteaux differentiable functional $J : H \to \mathbf{R}$, with $J(0) = 0$, such that $J' = T$, where J' is the Gâteaux derivative of J. The functional J is said to be the potential of T. It can easily be checked that

$$J(x) = \int_0^1 \langle T(sx), x \rangle ds$$

for all $x \in H$.

THEOREM 2 (B. R., 2012) . - *Let $T : H \to H$ be a nonexpansive potential operator and J its potential. For each $r > 0$, put*

$$\varphi(r) = \inf_{x \in B_r} \frac{\sup_{B_r} J - J(x)}{r - \|x\|^2} .$$

If there is $r > 0$ such that either $\varphi(r) < \frac{1}{2}$ or

$$\frac{\sup_{S_r} J}{r} < \sup_{\|x\|^2 > r} \frac{J(x)}{\|x\|^2} ,$$

then T has a fixed point which lies in B_r.
If $T(0) \neq 0$, then one has

$$\liminf_{r \to 0^+} \varphi(r) \geq \frac{1}{2} .$$

In any case, one has

$$\limsup_{r \to +\infty} \varphi(r) \leq \frac{1}{2} .$$

COROLLARY 1. - *If T has no fixed points, then*

$$\lim_{r \to +\infty} \varphi(r) = \inf_{r > 0} \varphi(r) = \frac{1}{2} \, .$$

THEOREM 3 (B. R., 2012) . - *Let* $T : H \to H$ *be a nonexpansive potential operator and* J *its potential. Assume that* $T(0) \neq 0$. *Set*

$$\theta = (\mathrm{dist}(0, \mathrm{Fix}(T)))^2$$

and

$$\psi(r) = \sup_{S_r} J$$

for all $r > 0$. *For each* $\lambda > 1$, *let* $\hat{u}_\lambda$ *be the unique fixed point of the operator* $\frac{1}{\lambda}T$.

Then, the following assertions hold:

(b_1) *the function* $\lambda \to h(\lambda) := \|\hat{u}_\lambda\|^2$ *is decreasing in* $]1, +\infty[$ *and its range is* $]0, \theta[$;

(b_2) *for each* $r \in]0, \theta[$, *the point* $\hat{v}_r := \hat{u}_{h^{-1}(r)}$ *is the unique global maximum of* $J_{|S_r}$ *towards which every maximizing sequence for* $J_{|S_r}$ *converges ;*

(b_3) *the function* $r \to \hat{v}_r$ *is continuous in* $]0, \theta[$.

If, in addition, the functional J is sequentially weakly continuous and has no local maxima in B_θ, then the following further assertions hold:

(b_4) *the function ψ is C^1, increasing and strictly concave in* $]0,\theta[$ *;*

(b_5) *one has*

$$T(\hat{v}_r) = 2\psi'(r)\hat{v}_r$$

for all $r \in]0,\theta[$;

(b_6) *one has*

$$\psi'(r) = \frac{1}{2}h^{-1}(r)$$

for all $r \in]0,\theta[$.

In the previous result, the essential assumption is that $T(0) \neq 0$. In the next result, to the contrary, we highlight a remarkable uniqueness property occurring when $\sup_X J = 0$ (and so $T(0) = 0$). Actually, in such a case, 0 is the unique fixed point of λT for each $\lambda \in]0, 3[$.

More precisely, for each real Hilbert space $(Y, \langle \cdot, \cdot \rangle)$, we denote by $\mathcal{A}_Y$ the class of all nonexpansive potential operators $P : Y \to Y$ such that

$$\sup_{x \in Y} \int_0^1 \langle P(sx), x \rangle ds = 0 \ .$$

Set

$$\gamma_Y = \inf_{P \in \mathcal{A}_Y} \inf\{\lambda > 0 : x = \lambda P(x) \ \text{for some} \ x \neq 0\} \ .$$

We have:

THEOREM 4 (B. R., 2012) . - *For any real Hilbert space* $(Y, \langle \cdot, \cdot \rangle)$, *with* $Y \neq \{0\}$, *one has*

$$\gamma_Y = 3 \; .$$

THEOREM 5 (B. R., 2014). - *Let H be an infinite-dimensional real Hilbert space and let $T : H \to H$ be a nonexpansive potential operator whose potential is sequentially weakly lower semicontinuous.*
Then, there exists a closed ball B in H such that $(\mathrm{id} + T)(B)$ intersects each convex dense subset of H.

REMARK 1. - Theorem 5 is no longer true if the potential of T is not sequentially weakly lower semicontinuous. In this connection, the simplest example is provided by $T(x) = -x$. Actually, since H is infinite-dimensional, the norm is not sequentially weakly upper semicontinuous.

REMARK 2. - Under the assumptions of Theorem 5, it may happen that the set $(\mathrm{id} + T)(B)$ has an empty interior for every ball B in H. In this connection, consider the case where T is a compact, symmetric, negative linear operator with norm 1. In such a case, by classical results, the potential of T is sequentially weakly continuous and $(\mathrm{id} + T)(H) \neq H$. Since $(\mathrm{id} + T)(H)$ is a linear subspace, this clearly implies that $\mathrm{int}(\mathrm{id} + T)(H)) = \emptyset$.

THEOREM 6. (J. Saint Raymond, 1984) - *Let E be a Banach space, let $X \subset E$ be a compact convex set and let $F : X \to 2^X$ be a continuous multifunction with non-empty closed convex values.*
Then, one has

$$\dim(\mathrm{Fix}(F)) \geq \inf_{x \in X} \dim((F(x)) \, .$$

THEOREM 7 (B. R., 1998). - *Let* E, F *be two Banach spaces,* $\Phi : E \to F$ *a continuous linear surjective operator and* $\Psi : E \to F$ *a continuous compact operator with bounded range. Then, the set* $\{x \in E : \Phi(x) = \Psi(x)\}$ *contains a compact set whose topological dimension is greater than or equal to* $\dim(\Phi^{-1}(0))$ *.*

EXAMPLE 1. - For every continuous and bounded functional $I : C^2([0, \pi]) \to \mathbf{R}$, the set

$$\{u \in C^2([0, \pi]) : u''(t) + u(t) = I(u) \cos t \; \forall t \in [0, \pi], u(0) = u(\pi) = 0\}$$

contains a non-degenerate connected set.

THEOREM 8 (J. Saint Raymond, 1994). - *Let X be a complete metric space and let $F : X \to 2^X$ be a multivalued contraction with compact values.*
Then, $\mathrm{Fix}(F)$ *is compact.*

THEOREM 9 (J. Saint Raymond, 1994). - *Let E be a Banach space, $X \subseteq E$ a closed convex set and $F : X \to 2^X$ a multivalued contraction with closed and unbounded values. Then,* $\mathrm{Fix}(F)$ *is unbounded.*

THEOREM 10 (B.R., 1987). - *Let E be a Banach space, $X \subseteq E$ a closed convex set and $F : X \to 2^X$ a multivalued contraction with closed convex values.*
Then, $\mathrm{Fix}(F)$ *is a retract of X.*

THEOREM 11 (B.R., 1987). - *Let E, F be two Banach spaces, $\Phi : E \to F$ a continuous linear surjective and non-injective operator, $\Psi : E \to F$ a Lipschitzian operator with Lipschitz constant $L < \frac{1}{\alpha}$, where*

$$\alpha := \sup_{\|y\|_F \leq 1} \operatorname{dist}(0, \Phi^{-1}(0)) .$$

Then, the multifunction $y \to (\Phi + \Psi)^{-1}(y)$ is Lipschitzian with Lipschitzian constant $\frac{\alpha}{1-\alpha L}$ and its values are unbounded retract of E.

THEOREM 12 (B.R., 2016). - *Let E be a Banach space, $\Phi : E \to \mathbf{R}$ a non-zero continuous linear functional, $\Psi : E \to \mathbf{R}$ a Lipschitzian functional with Lipschitz constant equal to $\|\Phi\|_{E^*}$. Then, one has*

$$\max\{\inf_X(\Phi+\Psi), \inf_X(\Phi-\Psi) = \inf_X(\Phi+|\Psi|) = \inf_X(\Phi+|\Psi|+\exp(-|\Psi|)) .$$

In particular, this implies that

$$\liminf_{\|x\|\to+\infty}(\Phi(x) + |\Psi(x)|) = \inf_X(\Phi + |\Psi|) .$$

THEOREM 13 (J. Saint Raymond, 2019). - *Let E be a normed space with $\dim(E) \geq 2$.*
Then, there exists a nonexpansive multifunction $F : E \to 2^E$, with closed convex values, such that $\mathrm{Fix}(F)$ is not connected.

References

[1] F. CAMMAROTO and A. CHINNÌ, *A complement to Ekeland's variational principle in Banach spaces*, Bull. Polish Acad. Sci. Math., **44** (1996), 29-33.

[2] B. RICCERI, *Une propriété topologique de l'ensemble des points fixes d'une contraction multivoques à valeurs convexes*, Rend. Accad. Naz. Lincei, (8) **81** (1987), 283-286.

[3] B. RICCERI, *Structure, approximation et dépendance continue des solutions de certaines équations non linéaires*, C. R. Acad. Sci. Paris, Série I, **305** (1987), 45-47.

[4] B. RICCERI, *On the topological dimension of the solution set of a class of nonlinear equations*, C. R. Acad. Sci. Paris, Série I, **325** (1997), 65-70.

[5] B. RICCERI, *Covering dimension and nonlinear equations*, RIMS, Kyoto, Surikai sekikenkyusho-Kokyuroku, **1031** (1998),

97-100.

[6] B. RICCERI, *Fixed points of nonexpansive potential operators in Hilbert spaces*, Fixed Point Theory Appl. **2012**, 2012: 123.

[7] B. RICCERI, *Another fixed point theorem for nonexpansive potential operators*, Studia Math., **211** (2012), 147-151.

[8] B. RICCERI, *A range property related to non-expansive operators*, Mathematika, **60** (2014), 232-236.

[9] B. RICCERI, *On the infimum of certain functionals*, in "Essays in Mathematics and its Applications - In Honor of Vladimir Arnold", Th. M. Rassias and P. M. Pardalos eds., 361-367, Springer, 2016.

[10] J. SAINT RAYMOND, *Points fixes des multiapplications à valeurs convexes*, C. R. Acad. Sci. Paris Sér. I Math., **298** (1984), 71-74.

[11] J. SAINT RAYMOND, *Multivalued contractions*, Set-Valued

Anal., **2** (1994), 559-571.

[12] J. SAINT RAYMOND, *Fixed points for non-expansive set-valued mappings*, Matematiche (Catania) **74** (2019), 321-336.