A survey of selected results in fixed point theory

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THEOREM 1 (F. Cammaroto and A. Chinnì , 1996) - Let $E$ be a Banach space, $f : E \to E$ and $\varphi : E \to [0, +\infty]$ a lower semicontinuous function such that

$$\|x - f(x)\| \leq \varphi(x) - \varphi(f(x))$$

for all $x \in X$. Moreover, assume that

$$\limsup_{\|x\| \to +\infty} \frac{\varphi(x)}{\|x\|} < 1 .$$

Then, the convex hull of $\text{Fix}(f)$ is dense in $X$. 


Let \((H, \langle \cdot, \cdot \rangle)\) will be a real Hilbert space.

For each \(r > 0\), we put

\[
B_r = \{ x \in H : \|x\|^2 < r \}
\]

and

\[
S_r = \{ x \in H : \|x\|^2 = r \}.
\]

A continuous operator \(T : H \to H\) is said to be a potential operator if there exists a Gâteaux differentiable functional \(J : H \to \mathbb{R}\), with \(J(0) = 0\), such that \(J' = T\), where \(J'\) is the Gâteaux derivative of \(J\). The functional \(J\) is said to be the potential of \(T\). It can easily be checked that

\[
J(x) = \int_0^1 \langle T(sx), x \rangle ds
\]

for all \(x \in H\).
THEOREM 2 (B. R., 2012) - Let $T : H \rightarrow H$ be a nonexpansive potential operator and $J$ its potential. For each $r > 0$, put

$$\varphi(r) = \inf_{x \in B_r} \frac{\sup_{B_r} J - J(x)}{r - \norm{x}^2}.$$ 

If there is $r > 0$ such that either $\varphi(r) < \frac{1}{2}$ or

$$\frac{\sup_{S_r} J}{r} < \sup_{\norm{x}^2 > r} \frac{J(x)}{\norm{x}^2},$$

then $T$ has a fixed point which lies in $B_r$.

If $T(0) \neq 0$, then one has

$$\liminf_{r \to 0^+} \varphi(r) \geq \frac{1}{2}.$$ 

In any case, one has

$$\limsup_{r \to +\infty} \varphi(r) \leq \frac{1}{2}.$$
COROLLARY 1. - If $T$ has no fixed points, then

\[
\lim_{r \to +\infty} \varphi(r) = \inf_{r > 0} \varphi(r) = \frac{1}{2}.
\]
THEOREM 3 (B. R., 2012). - Let $T : H \to H$ be a nonexpansive potential operator and $J$ its potential. Assume that $T(0) \neq 0$. Set

$$\theta = (\text{dist}(0, \text{Fix}(T)))^2$$

and

$$\psi(r) = \sup_{S_r} J$$

for all $r > 0$. For each $\lambda > 1$, let $\hat{u}_\lambda$ be the unique fixed point of the operator $\frac{1}{\lambda} T$.

Then, the following assertions hold:

$(b_1)$ the function $\lambda \to h(\lambda) := \|\hat{u}_\lambda\|^2$ is decreasing in $]1, +\infty[$ and its range is $]0, \theta[$;

$(b_2)$ for each $r \in ]0, \theta[$, the point $\hat{v}_r := \hat{u}_{h^{-1}(r)}$ is the unique global maximum of $J|_{S_r}$ towards which every maximizing sequence for $J|_{S_r}$ converges;

$(b_3)$ the function $r \to \hat{v}_r$ is continuous in $]0, \theta[$.
If, in addition, the functional $J$ is sequentially weakly continuous and has no local maxima in $B_\theta$, then the following further assertions hold:

$(b_4)$ the function $\psi$ is $C^1$, increasing and strictly concave in $]0, \theta[$;

$(b_5)$ one has

$$T(\hat{v}_r) = 2\psi'(r)\hat{v}_r$$

for all $r \in ]0, \theta[$;

$(b_6)$ one has

$$\psi'(r) = \frac{1}{2}h^{-1}(r)$$

for all $r \in ]0, \theta[$.
In the previous result, the essential assumption is that \( T(0) \neq 0 \). In the next result, to the contrary, we highlight a remarkable uniqueness property occurring when \( \sup_X J = 0 \) (and so \( T(0) = 0 \)). Actually, in such a case, 0 is the unique fixed point of \( \lambda T \) for each \( \lambda \in ]0, 3[ \).

More precisely, for each real Hilbert space \((Y, \langle \cdot, \cdot \rangle)\), we denote by \( A_Y \) the class of all nonexpansive potential operators \( P : Y \to Y \) such that

\[
\sup_{x \in Y} \int_0^1 \langle P(sx), x \rangle \, ds = 0 .
\]

Set

\[
\gamma_Y = \inf_{P \in A_Y} \inf \{ \lambda > 0 : x = \lambda P(x) \text{ for some } x \neq 0 \} .
\]

We have:
THEOREM 4 (B. R., 2012). - For any real Hilbert space \((Y, \langle \cdot, \cdot \rangle)\), with \(Y \neq \{0\}\), one has

\[ \gamma_Y = 3. \]
THEOREM 5 (B. R., 2014). - Let $H$ be an infinite-dimensional real Hilbert space and let $T : H \to H$ be a nonexpansive potential operator whose potential is sequentially weakly lower semicontinuous.

Then, there exists a closed ball $B$ in $H$ such that $(\text{id} + T)(B)$ intersects each convex dense subset of $H$.

REMARK 1. - Theorem 5 is no longer true if the potential of $T$ is not sequentially weakly lower semicontinuous. In this connection, the simplest example is provided by $T(x) = -x$. Actually, since $H$ is infinite-dimensional, the norm is not sequentially weakly upper semicontinuous.

REMARK 2. - Under the assumptions of Theorem 5, it may happen that the set $(\text{id} + T)(B)$ has an empty interior for every ball $B$ in $H$. In this connection, consider the case where $T$ is a compact, symmetric, negative linear operator with norm 1. In such a case, by classical results, the potential of $T$ is
sequentially weakly continuous and \((\text{id} + T)(H) \neq H\). Since \((\text{id} + T)(H)\) is a linear subspace, this clearly implies that 
\[ \text{int}(\text{id} + T)(H) = \emptyset. \]
THEOREM 6. (J. Saint Raymond, 1984) - Let $E$ be a Banach space, let $X \subset E$ be a compact convex set and let $F : X \to 2^X$ be a continuous multifunction with non-empty closed convex values.
Then, one has
\[
\dim(\text{Fix}(F)) \geq \inf_{x \in X} \dim((F(x))).
\]
THEOREM 7 (B. R., 1998). - Let $E, F$ be two Banach spaces, $\Phi : E \to F$ a continuous linear surjective operator and $\Psi : E \to F$ a continuous compact operator with bounded range. Then, the set $\{x \in E : \Phi(x) = \Psi(x)\}$ contains a compact set whose topological dimension is greater than or equal to $\dim(\Phi^{-1}(0))$. 
EXAMPLE 1. - For every continuous and bounded functional $l : C^2([0, \pi]) \to \mathbb{R}$, the set

$$\{ u \in C^2([0, \pi]) : u''(t) + u(t) = l(u) \cos t \ \forall t \in [0, \pi], \ u(0) = u(\pi) = 0 \}$$

contains a non-degenerate connected set.
THEOREM 8 (J. Saint Raymond, 1994). - Let $X$ be a complete metric space and let $F : X \to 2^X$ be a multivalued contraction with compact values.

Then, $\text{Fix}(F)$ is compact.
THEOREM 9 (J. Saint Raymond, 1994). - Let $E$ be a Banach space, $X \subseteq E$ a closed convex set and $F : X \to 2^X$ a multivalued contraction with closed and unbounded values. Then, $\text{Fix}(F)$ is unbounded.
THEOREM 10 (B.R., 1987). - Let $E$ be a Banach space, $X \subseteq E$ a closed convex set and $F : X \to 2^X$ a multivalued contraction with closed convex values.
Then, $\text{Fix}(F)$ is a retract of $X$. 
THEOREM 11 (B.R., 1987). - Let $E, F$ be two Banach spaces, $\Phi : E \to F$ a continuous linear surjective and non-injective operator, $\Psi : E \to F$ a Lipschitzian operator with Lipschitz constant $L < \frac{1}{\alpha}$, where

$$\alpha := \sup_{\|y\|_F \leq 1} \operatorname{dist}(0, \Phi^{-1}(0)).$$

Then, the multifunction $y \mapsto (\Phi + \Psi)^{-1}(y)$ is Lipschitzian with Lipschitzian constant $\frac{\alpha}{1 - \alpha L}$ and its values are unbounded retract of $E$. 

THEOREM 12 (B.R., 2016). - Let $E$ be a Banach space, $\Phi : E \to \mathbb{R}$ a non-zero continuous linear functional, $\Psi : E \to \mathbb{R}$ a Lipschitzian functional with Lipschitz constant equal to $\|\Phi\|_{E^*}$. Then, one has

$$\max\{\inf_X (\Phi + \Psi), \inf_X (\Phi - \Psi) = \inf_X (\Phi + |\Psi|) = \inf_X (\Phi + |\Psi| + \exp(-|\Psi|))\}.$$ 

In particular, this implies that

$$\liminf_{\|x\| \to +\infty} (\Phi(x) + |\Psi(x)|) = \inf_X (\Phi + |\Psi|).$$
THEOREM 13 (J. Saint Raymond, 2019). - Let $E$ be a normed space with $\dim(E) \geq 2$. Then, there exists a nonexpansive multifunction $F : E \rightarrow 2^E$, with closed convex values, such that $\text{Fix}(F)$ is not connected.
References


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