Local linear convergence of alternating projections in metric spaces with bounded curvature

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A workshop on Nonlinear Functional Analysis and Its Applications
in memory of Professor Ronald E. Bruck
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A.S. Lewis, G. López-Acedo, A. Nicolae, Local linear convergence of alternating projections in metric spaces with bounded curvature, SIAM J. Optim, in press.

Aim of this talk

- to give a brief history of the method of alternating projections (MAP) employed to find a point in the intersection of two closed sets.
- to sketch the proof of linear convergence of MAP (in absence of convexity) using two local main ingredients:

  super-regularity and separable intersection.

- to study these two properties and their relation to other relevant properties.

Context: Alexandrov spaces, although for simplicity, some properties and arguments will be given in $\mathbb{R}^n$.

Alternating projections

X metric space, $A, B \subseteq X$ closed, $A \cap B \neq \emptyset$
A sequence $(x_n) \subseteq X$ is an alternating projection sequence starting at $x_0 \in X$: $\forall n \in \mathbb{N}$,

$$x_{2n+1} \in P_B(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A(x_{2n+1}).$$

Rate of convergence

A sequence $(x_n) \subseteq X$ that converges to $x \in X$:

- converges linearly to x: $\exists$ a constant $k > 0$ and a rate $a \in (0, 1)$ s.t.

$$\forall n \in \mathbb{N} \, (d(x_n, x) \leq k \cdot a^n) \, .$$

- has a rate of convergence $\alpha : (0, \infty) \to \mathbb{N}$:

$$\forall \varepsilon > 0 \, \forall n \geq \alpha(\varepsilon) \; (d(x_n, x) < \varepsilon) \, .$$

Any linearly convergent sequence has a rate of convergence that is logarithmic in $1/\varepsilon$ for $\varepsilon < k$.

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Convex case - Global convergence

H Hilbert space, $A, B \subseteq H$ closed, $A \cap B \neq \emptyset$

- von Neumann 1933: A, B subspaces.
- Bregman 1965: A, B convex – weak convergence.
- Hundal 2004: in general only weak convergence for closed convex sets.

Key facts:

- P_A, P_B singlevalued and firmly nonexpansive.
- (x_n) is Fejér monotone w.r.t. $A \cap B$.
- can be slow converging even in $\mathbb{R}^n$.
- R.E Bruck: the rate of asymptotic regularity.

More recently: results in sufficiently regular geodesic spaces.

- Bacak, X CAT(0).
- Aríza,López and Nicolae , X CAT(κ) , $\kappa > 0$.

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Nonconvex case - Local convergence

$A, B \subseteq \mathbb{R}^n$ closed, $z \in A \cap B \neq \emptyset$

- P_A, P_B multivalued and global convergence observed heuristically for some scenarios – very limited mathematical foundation.
- local linear convergence in the presence of appropriate local geometric features: e.g. transversality: $N_A(z) \cap (-N_B(z)) = \{0\}$, $N_A(z)$: points near z projected onto $A \to$ vectors $\to$ limits when approaching z.

D. Drusvyatskiy, A.D. Ioffe, A.S. Lewis, Transversality and alternating projections for nonconvex sets, Found. Comput. Math. 15 (2015), 1637–1651.

Alexandrov angle

X geodesic space
$\gamma : [0,l] \to X$, $\gamma' : [0,l'] \to X$ nonconstant geodesics with $\gamma(0) = \gamma'(0)$

$t \in (0,l]$, $t' \in (0,l']$
$\overline{\angle}_{\gamma(0)}(\gamma(t), \gamma'(t'))$ – interior angle at $\overline{\gamma(0)}$ in $\Delta(\overline{\gamma(0)}, \overline{\gamma(t)}, \overline{\gamma'(t')}) \subseteq \mathbb{R}^2$

Alexandrov angle between γ and γ':

$$\angle(\gamma, \gamma') = \lim_{\varepsilon \searrow 0} \sup_{0<t,t'<\varepsilon} \overline{\angle}_{\gamma(0)}(\gamma(t), \gamma'(t')) \in [0, \pi].$$

Notation $\angle_x(y,z)$ if unique geodesics from x to y and x to z.

Super-regularity

Definition

Let (X, d) be a geodesic space and $A \subseteq X$. We say that A is *super-regular* at $z \in A$ if given any $\varepsilon > 0$ there exists $r > 0$ such that any two points in $B(z, r)$ are joined by a unique geodesic segment and for all $y \in B(z, r/2) \setminus A$, $x \in P_A(y)$, and all $x' \in A \cap B(z, r)$ with $x' \neq x$, $\angle_x(y, x') \geq \pi/2 - \varepsilon$.

A.S. Lewis, D.R. Luke, J. Malick, Local linear convergence for alternating and averaged nonconvex projections, Found. Comput. Math. 9 (2009), 485–513.

Separable intersection

Definition

Let (X, d) be a geodesic space and $A, B \subseteq X$. We say that A *intersects* B *separably* at $z \in A \cap B$ if there exist $\alpha, r > 0$ such that any two points in $B(z, r)$ are joined by a unique geodesic segment and for all $x \in (A \cap B(z, r)) \setminus B$, $y \in P_B(x) \setminus A$ satisfying $\max\{d(y, x), d(y, z)\} < r/2$ and all $x' \in P_A(y)$, $\angle_y(x, x') \geq \alpha$.

D. Noll, A. Rondepierre, On local convergence of the method of alternating projections, Found. Comput. Math. 16 (2016), 425–455.

Main result

Theorem

Let $\kappa > 0$, X *be a complete* $\mathrm{CAT}(\kappa)$ *space, and* A, B *be closed subsets of* X. *Suppose that, at* $z \in A \cap B$, A *is super-regular and intersects* B *separably. Then any alternating projection sequence* (x_n) *starting at* $x_0 \in A$ *sufficiently close to* z *converges linearly to a point in* $A \cap B$.

Local linear convergence of MAP (Sketch)

$A, B \subseteq \mathbb{R}^n$ closed, $z \in A \cap B$, (x_n) alternating projection sequence, $\varepsilon > 0$:

- A super-regular at $z \implies \angle_{x_{2n+2}}(x_{2n}, x_{2n+1}) \geq \pi/2 - \varepsilon$.
- A intersects B separably at $z \implies \angle_{x_{2n+1}}(x_{2n}, x_{2n+2}) \geq \alpha$.

Hence,

- $\angle_{x_{2n}}(x_{2n+1}, x_{2n+2}) \leq \pi/2 + \varepsilon - \alpha$.
- $\dfrac{\|x_{2n+2} - x_{2n+1}\|}{\sin(\pi/2 + \varepsilon - \alpha)} \leq \dfrac{\|x_{2n} - x_{2n+1}\|}{\sin(\pi/2 - \varepsilon)}$.
- for fixed $c \in (\cos\alpha, 1)$, taking ε sufficiently small,

$$\|x_{2n+2} - x_{2n+3}\| \leq \|x_{2n+2} - x_{2n+1}\| \leq c\|x_{2n} - x_{2n+1}\|.$$

- (x_n) converges linearly to a point in $A \cap B$ with rate $\sqrt{c}$.

A.S. Lewis, D.R. Luke, J. Malick, Local linear convergence for alternating and averaged nonconvex projections, Found. Comput. Math. 9 (2009), 485–513.

D. Drusvyatskiy, A.S. Lewis, Local linear convergence for inexact alternating projections on nonconvex sets, Vietnam J. Math. 47 (2019), 669–681.

Uniform approximation by geodesics (UAG)

(X, d) metric space, $A \subseteq X$ is uniformly approximable by geodesics (UAG) at $z \in A$: $\forall \varepsilon > 0$, if $x, x' \in A$ are distinct and sufficiently close to z, there exists a mapping $f : [0, l] \to A$ with $f(0) = x$ and $f(l) = x'$, and a geodesic $\gamma : [0, l] \to X$ geodesic starting at x with

$$\frac{d(\gamma(t), f(t))}{t} < \varepsilon \quad \forall t \in (0, l].$$

$A \subseteq \mathbb{R}^n$ is UAG at $z \in A$: $\forall \varepsilon > 0$, if $x, x' \in A$ are distinct and sufficiently close to z, there exists a mapping $f : [0, 1] \to A$ with $f(0) = x$ and $f(1) = x'$, and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$\|f(t) - (x + td)\| \leq \varepsilon t \|d\| \quad \forall t \in [0, 1].$$

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Super-regularity via UAG

Theorem

X $\mathrm{CAT}(\kappa)$ space, A UAG at $z \Longrightarrow A$ super-regular at z.

UAG and super-regularity are not persistent nearby.

Define $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{2^n}\left(x - \frac{1}{2^{n+1}}\right) & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}\right], n \in \mathbb{N} \\ \frac{1}{2^n}\left(\frac{1}{2^n} - x\right) & \text{if } x \in \left(\frac{3}{2^{n+2}}, \frac{1}{2^n}\right], n \in \mathbb{N} \\ 0 & \text{if } x = 0. \end{cases}$$

$A = \operatorname{gph} f$, A is UAG at $(0,0)$, but is not UAG at all points in a ball centered at $(0,0)$.

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Where does UAG hold?

(X, d) metric space, $A \subseteq X$, $z \in A$

A has a finite extrinsic curvature at z: $\exists \sigma \geq 0, \exists r > 0$ s.t.

$$d^A(p,q) - d(p,q) \leq \sigma\, d(p,q)^3 \quad \forall p, q \in B(z,r) \cap A.$$

- based on Haantjes notion for curves.
- local version of the notion of 2-convexity (Lytchak).

X $\mathrm{CAT}(\kappa)$ space, $A \subseteq X$ locally compact
A finite extrinsic curvature at $z \Longrightarrow A$ UAG at z.

Haantjes, Distance geometry. Curvature in abstract metric spaces, Nederl. Akad. Wetensch., Proc. 50 (1947), 496–508.

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$A \subseteq \mathbb{R}^n$ closed, $z \in A$

A is prox-regular at z: $\exists r > 0$ s.t. $\operatorname{dist}(\cdot, A)$ is continuously diferentiable on $B(z, r) \setminus A$.

A has positive reach: $\exists \delta > 0$ s.t. $\forall x \in X$ with $\operatorname{dist}(x, A) < \delta$, $P_A(x)$ is a singleton.

A is prox-regular at z

$\Updownarrow$

$\exists R > 0$ s.t. $A \cap \overline{B}(z, R)$ has positive reach

$\Updownarrow$

$\exists R > 0$ s.t. $A \cap \overline{B}(z, R)$ is 2-convex

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A has a finite extrinsic curvature at z

$\Downarrow \not\Uparrow$

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A is UAG at z

What other sets are UAG?

Images of convex sets under a sufficiently smooth function (Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on a neighborhood of u, DF is continuous at u, and $DF(u)$ is injective. Then for any sufficiently small $r > 0$, $F(B(u,r) \cap C)$ is UAG at $F(u)$).

Epigraphs of approximately convex functions:
$f : \mathbb{R}^n \to (-\infty, \infty]$ is approximately convex at $z \in \mathbb{R}^n$:
$\forall \varepsilon > 0$, $\forall x, x'$ sufficiently close to z,

$$f((1-t)x + tx') \leq (1-t)f(x) + tf(x') + \varepsilon t(1-t)\|x' - x\|, \quad \forall t \in [0,1].$$

Sets defined by C^1 inequality constraints satisfying the Mangasarian-Fromovitz condition:
$A \subseteq \mathbb{R}^n$, U neighborhood of $z \in A$, $f_1, \dots, f_m : U \to \mathbb{R}$ C^1 functions s.t.

$$A \cap U = \{x \in U \mid f_j(x) \leq 0, \forall j \in \{1, \dots, m\}\}$$

and $\exists d \in \mathbb{R}^n$ s.t. $Df_j(z)(d) < 0, \forall j \in \{1, \dots, m\}$.

H.V. Ngai, D.T. Luc, M. Théra, Approximate convex functions, J. Nonlinear Convex Anal. 1 (2000), 155–176.

R. T. Rockafellar, R. J. B. Wets, Variational analysis, Springer-Verlag, Berlin, 1998.

Separable intersection

X $\mathrm{CAT}(\kappa)$ space, $A, B \subseteq X$, $z \in A \cap B$, A, B both super-regular at z

$\exists \alpha > 0$ s.t. if $p \in A \setminus B$ and $q \in B \setminus A$ are sufficiently close to z, then $\angle_z(p, q) \geq \alpha$
$\Longrightarrow$ A intersects B separably at z.

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$\implies A$ intersects B separably at z.

Separable intersection - convex case

X complete $\mathrm{CAT}(\kappa)$ space, r-geodesic extension property
$A, B \subseteq X$ closed convex, $z \in A \cap B$

A and B are transversal at z: $\nexists\ \gamma : [0, l] \to X$ nonconstant geodesic s.t.
$z = \gamma(l/2)$ and $z \in P_A(\gamma(0)) \cap P_B(\gamma(l))$.

X additionally locally compact
A, B transversal at $z \implies A$ intersects B separably at z.

X additionally $\mathrm{CBB}(\kappa')$
$\exists w \in \mathrm{int}(A) \cap B$ with $d(w, z) < r \implies A, B$ transversal at z.

Example

Take a unit vector a and the set $A = \{x \in \mathbb{S}^n \mid \langle x, a \rangle = 1/2\}$. This set is nonconvex but super-regular at all its points.

Take another unit vector b such that $\langle a, b \rangle \in (-\sqrt{3}/2, 0) \cup (0, \sqrt{3}/2)$, and let B be the subset of $\mathbb{S}^n$ orthogonal to b. The sets A and B will intersect, and we fix $z \in A \cap B$. Note that B is super-regular at z since it is weakly convex. Moreover, A and B intersect separably there.