Local linear convergence of alternating projections in metric spaces with bounded curvature

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A workshop on Nonlinear Functional Analysis and Its Applications in memory of Professor Ronald E. Bruck
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Aim of this talk

- to give a brief history of the method of alternating projections (MAP) employed to find a point in the intersection of two closed sets.

- to sketch the proof of linear convergence of MAP (in absence of convexity) using two local main ingredients:
  
  super-regularity and separable intersection.

- to study these two properties and their relation to other relevant properties.

Context: Alexandrov spaces, although for simplicity, some properties and arguments will be given in $\mathbb{R}^n$. 
Alternating projections

Let $X$ be a metric space, $A, B \subseteq X$ closed, $A \cap B \neq \emptyset$.

A sequence $(x_n) \subseteq X$ is an alternating projection sequence starting at $x_0 \in X$: \forall n \in \mathbb{N},

$$x_{2n+1} \in P_B(x_{2n}) \quad \text{and} \quad x_{2n+2} \in P_A(x_{2n+1}).$$
A sequence \((x_n) \subseteq X\) that converges to \(x \in X\):

- **converges linearly** to \(x\): \(\exists\) a constant \(k > 0\) and a rate \(a \in (0, 1)\) s.t.

\[
\forall n \in \mathbb{N} \ (d(x_n, x) \leq k \cdot a^n).
\]

- has a rate of convergence \(\alpha : (0, \infty) \rightarrow \mathbb{N}\):

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\forall \varepsilon > 0 \forall n \geq \alpha(\varepsilon) \ (d(x_n, x) < \varepsilon).
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Any linearly convergent sequence has a rate of convergence that is logarithmic in \(1/\varepsilon\) for \(\varepsilon < k\).
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Convex case - Global convergence

\[ H \text{ Hilbert space, } A, B \subseteq H \text{ closed, } A \cap B \neq \emptyset \]

- von Neumann 1933: \( A, B \) subspaces.
- Hundal 2004: in general only weak convergence for closed convex sets.

Key facts:

- \( P_A, P_B \) singlevalued and firmly nonexpansive.
- \( (x_n) \) is Fejér monotone w.r.t. \( A \cap B \).
- can be slow converging even in \( \mathbb{R}^n \).
- R.E Bruck: the rate of asymptotic regularity.

More recently: results in sufficiently regular geodesic spaces.

- Bacak, \( X \) CAT(0).
- Aríza, López and Nicolae, \( X \) CAT(\( \kappa \)), \( \kappa > 0 \).
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Nonconvex case - Local convergence

\(A, B \subseteq \mathbb{R}^n\) closed, \(z \in A \cap B \neq \emptyset\)

- \(P_A, P_B\) multivalued and global convergence observed heuristically for some scenarios – very limited mathematical foundation.
- local linear convergence in the presence of appropriate local geometric features: e.g. transversality: \(N_A(z) \cap (-N_B(z)) = \{0\}\), \(N_A(z)\): points near \(z\) projected onto \(A \rightarrow\) vectors \(\rightarrow\) limits when approaching \(z\).

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Alexandrov angle

$X$ geodesic space
$\gamma : [0, l] \to X$, $\gamma' : [0, l'] \to X$ nonconstant geodesics with $\gamma(0) = \gamma'(0)$
$t \in (0, l]$, $t' \in (0, l']$
$\angle_{\gamma(0)} (\gamma(t), \gamma'(t'))$ – interior angle at $\gamma(0)$ in $\Delta(\gamma(0), \gamma(t), \gamma'(t')) \subseteq \mathbb{R}^2$

**Alexandrov angle** between $\gamma$ and $\gamma'$:

$$\angle(\gamma, \gamma') = \lim_{\varepsilon \searrow 0} \sup_{0 < t, t' < \varepsilon} \angle_{\gamma(0)} (\gamma(t), \gamma'(t')) \in [0, \pi].$$

Notation $\angle_x(y, z)$ if unique geodesics from $x$ to $y$ and $x$ to $z$. 
Super-regularity

Definition

Let \((X, d)\) be a geodesic space and \(A \subseteq X\). We say that \(A\) is super-regular at \(z \in A\) if given any \(\varepsilon > 0\) there exists \(r > 0\) such that any two points in \(B(z, r)\) are joined by a unique geodesic segment and for all \(y \in B(z, r/2) \setminus A\), \(x \in P_A(y)\), and all \(x' \in A \cap B(z, r)\) with \(x' \neq x\), \(\angle_x(y, x') \geq \pi/2 - \varepsilon\).

Separable intersection

**Definition**

Let \((X, d)\) be a geodesic space and \(A, B \subseteq X\). We say that \(A\) intersects \(B\) *separably* at \(z \in A \cap B\) if there exist \(\alpha, r > 0\) such that any two points in \(B(z, r)\) are joined by a unique geodesic segment and for all \(x \in (A \cap B(z, r)) \setminus B\), \(y \in P_B(x) \setminus A\) satisfying
\[
\max\{d(y, x), d(y, z)\} < r/2 \quad \text{and all} \quad x' \in P_A(y), \quad \angle_y(x, x') \geq \alpha.
\]

---

Main result

**Theorem**

Let $\kappa > 0$, $X$ be a complete CAT($\kappa$) space, and $A, B$ be closed subsets of $X$. Suppose that, at $z \in A \cap B$, $A$ is super-regular and intersects $B$ separably. Then any alternating projection sequence $(x_n)$ starting at $x_0 \in A$ sufficiently close to $z$ converges linearly to a point in $A \cap B$. 
Local linear convergence of MAP (Sketch)

\( A, B \subseteq \mathbb{R}^n \) closed, \( z \in A \cap B \), \( (x_n) \) alternating projection sequence, \( \varepsilon > 0 \):

- \( A \) super-regular at \( z \) \( \implies \angle x_{2n+2}(x_{2n}, x_{2n+1}) \geq \pi/2 - \varepsilon \).
- \( A \) intersects \( B \) separably at \( z \) \( \implies \angle x_{2n+1}(x_{2n}, x_{2n+2}) \geq \alpha \).

Hence,

- \( \angle x_{2n}(x_{2n+1}, x_{2n+2}) \leq \pi/2 + \varepsilon - \alpha \).
- \( \frac{\|x_{2n+2} - x_{2n+1}\|}{\sin(\pi/2 + \varepsilon - \alpha)} \leq \frac{\|x_{2n} - x_{2n+1}\|}{\sin(\pi/2 - \varepsilon)} \).

for fixed \( c \in (\cos \alpha, 1) \), taking \( \varepsilon \) sufficiently small,

\( \|x_{2n+2} - x_{2n+3}\| \leq \|x_{2n+2} - x_{2n+1}\| \leq c \|x_{2n} - x_{2n+1}\| \).

- \( (x_n) \) converges linearly to a point in \( A \cap B \) with rate \( \sqrt{c} \).

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Uniform approximation by geodesics (UAG) on $(X, d)$ metric space, $A \subseteq X$ is uniformly approximable by geodesics (UAG) at $z \in A$: $\forall \varepsilon > 0$, if $x, x' \in A$ are distinct and sufficiently close to $z$, there exists a mapping $f : [0, l] \rightarrow A$ with $f(0) = x$ and $f(l) = x'$, and a geodesic $\gamma : [0, l] \rightarrow X$ geodesic starting at $x$ with

$$\frac{d(\gamma(t), f(t))}{t} < \varepsilon \quad \forall t \in (0, l].$$

A $A \subseteq \mathbb{R}^n$ is UAG at $z \in A$: $\forall \varepsilon > 0$, if $x, x' \in A$ are distinct and sufficiently close to $z$, there exists a mapping $f : [0, 1] \rightarrow A$ with $f(0) = x$ and $f(1) = x'$, and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$\|f(t) - (x + td)\| \leq \varepsilon t \|d\| \quad \forall t \in [0, 1].$$

---

Super-regularity via UAG

**Theorem**

\[ X \text{ CAT}(\kappa) \text{ space, } A \text{ UAG at } z \implies A \text{ super-regular at } z. \]

UAG and super-regularity are not persistent nearby.

Define \( f : [0, 1] \to \mathbb{R}, \)

\[
f(x) = \begin{cases} 
\frac{1}{2^n} \left(x - \frac{1}{2^{n+1}}\right) & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}\right], n \in \mathbb{N} \\
\frac{1}{2^n} \left(\frac{1}{2^n} - x\right) & \text{if } x \in \left(\frac{3}{2^{n+2}}, \frac{1}{2^n}\right], n \in \mathbb{N} \\
0 & \text{if } x = 0.
\end{cases}
\]

\( A = \text{gph } f, \) A is UAG at \((0, 0), \) but is not UAG at all points in a ball centered at \((0, 0). \)
Super-regularity via UAG

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Where does UAG hold?

\((X, d)\) metric space, \(A \subseteq X, \ z \in A\)

\(A\) has a \textbf{finite extrinsic curvature} at \(z\): \(\exists \sigma \geq 0, \exists r > 0\ \text{s.t.}\)

\[d^A(p, q) - d(p, q) \leq \sigma d(p, q)^3\quad \forall p, q \in B(z, r) \cap A.\]

- based on Haantjes notion for curves.
- local version of the notion of 2-convexity (Lytchak).

\(X\) \(\text{CAT}(\kappa)\) space, \(A \subseteq X\) locally compact
\(A\) \text{finite extrinsic curvature at } z \quad \implies A \text{ UAG at } z.

\begin{itemize}
  \item A. Lytchak, Almost convex subsets, Geom. Dedicata 115 (2005), 201–218.
\end{itemize}
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\(X\) CAT(\(\kappa\)) space, \(A \subseteq X\) locally compact

\(A\) finite extrinsic curvature at \(z\) \(\implies\) \(A\) UAG at \(z\).

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Where does UAG hold?

\( A \subseteq \mathbb{R}^n \) closed, \( z \in A \)

\( A \) is prox-regular at \( z \): \( \exists r > 0 \) s.t. \( \text{dist}(\cdot, A) \) is continuously differentiable on \( B(z, r) \setminus A \).

\( A \) has positive reach: \( \exists \delta > 0 \) s.t. \( \forall x \in X \) with \( \text{dist}(x, A) < \delta \), \( P_A(x) \) is a singleton.

\[
\begin{align*}
A \text{ is prox-regular at } z & \quad \updownarrow \\
\exists R > 0 \text{ s.t. } A \cap \overline{B}(z, R) \text{ has positive reach} & \quad \updownarrow \\
\exists R > 0 \text{ s.t. } A \cap \overline{B}(z, R) \text{ is 2-convex} & \quad \updownarrow \\
A \text{ has a finite extrinsic curvature at } z & \quad \updownarrow \uparrow \\
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Where does UAG hold?

$A \subseteq \mathbb{R}^n$ closed, $z \in A$

$A$ is prox-regular at $z$: $\exists r > 0$ s.t. $\text{dist}(\cdot, A)$ is continuously differentiable on $B(z, r) \setminus A$.

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$A$ is prox-regular at $z$

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$\exists R > 0$ s.t. $A \cap \overline{B}(z, R)$ is 2-convex

$\Updownarrow$

$A$ has a finite extrinsic curvature at $z$

$\Downarrow \Uparrow$

$A$ is UAG at $z$
Images of convex sets under a sufficiently smooth function (Suppose that $F: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on a neighborhood of $u$, $DF$ is continuous at $u$, and $DF(u)$ is injective. Then for any sufficiently small $r > 0$, $F(B(u,r) \cap C)$ is UAG at $F(u)$).

Epigraphs of approximately convex functions:
$f: \mathbb{R}^n \to (-\infty, \infty]$ is approximately convex at $z \in \mathbb{R}^n$:
$\forall \varepsilon > 0$, $\forall x, x'$ sufficiently close to $z$,

$$f((1-t)x + tx') \leq (1-t)f(x) + tf(x') + \varepsilon t(1-t)\|x' - x\|, \quad \forall t \in [0,1].$$

Sets defined by $C^1$ inequality constraints satisfying the Mangasarian-Fromovitz condition:
$A \subseteq \mathbb{R}^n$, $U$ neighborhood of $z \in A$, $f_1, \ldots, f_m: U \to \mathbb{R} C^1$ functions s.t.

$$A \cap U = \{x \in U \mid f_j(x) \leq 0, \forall j \in \{1, \ldots, m\}\}$$

and $\exists d \in \mathbb{R}^n$ s.t. $Df_j(z)(d) < 0, \forall j \in \{1, \ldots, m\}$.


$X$ CAT($\kappa$) space, $A, B \subseteq X$, $z \in A \cap B$, $A, B$ both super-regular at $z$

$\exists \alpha > 0$ s.t. if $p \in A \setminus B$ and $q \in B \setminus A$ are sufficiently close to $z$, then
$
\angle_z (p, q) \geq \alpha
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$\implies A$ intersects $B$ separably at $z$. 
$X$ CAT$(\kappa)$ space, $A, B \subseteq X$, $z \in A \cap B$, $A, B$ both super-regular at $z$

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$\angle_z(p, q) \geq \alpha$

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$X$ complete CAT($\kappa$) space, $r$-geodesic extension property
$A, B \subseteq X$ closed convex, $z \in A \cap B$

$A$ and $B$ are transversal at $z$: $\not\exists \gamma : [0, l] \to X$ nonconstant geodesic s.t.
$z = \gamma(l/2)$ and $z \in P_A(\gamma(0)) \cap P_B(\gamma(l))$.

$X$ additionally locally compact
$A, B$ transversal at $z \implies A$ intersects $B$ separably at $z$.

$X$ additionally CBB($\kappa'$)
$\exists w \in \text{int}(A) \cap B$ with $d(w, z) < r \implies A, B$ transversal at $z$. 
Example

Take a unit vector $a$ and the set $A = \{x \in S^n \mid \langle x, a \rangle = 1/2\}$. This set is nonconvex but super-regular at all its points.

Take another unit vector $b$ such that $\langle a, b \rangle \in (-\sqrt{3}/2, 0) \cup (0, \sqrt{3}/2)$, and let $B$ be the subset of $S^n$ orthogonal to $b$. The sets $A$ and $B$ will intersect, and we fix $z \in A \cap B$. Note that $B$ is super-regular at $z$ since it is weakly convex. Moreover, $A$ and $B$ intersect separably there.