Smoothness of norms and the geometry of Banach spaces

Gilles Godefroy

Institut de Mathématiques de
Jussieu Paris Rive Gauche
(CNRS & UPMC-Université Paris-06)

Technion, 5 April 2022

One-sided differentiability.

Let X be a Banach space, equipped with its norm $\| \, . \, \|$. The norm is in particular a continuous convex function, hence it is one-sided directionally differentiable at every point. Let us investigate which uniformities can take place in this one-sided differentiability. Let $S_X = \{x \in X; \ \|x\| = 1\}$ be the unit sphere of X, and $x \in S_X$. We set

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t} = \nabla^+(x, h).$$

One-sided differentiability.

Let X be a Banach space, equipped with its norm $\| \, . \, \|$. The norm is in particular a continuous convex function, hence it is one-sided directionally differentiable at every point. Let us investigate which uniformities can take place in this one-sided differentiability. Let $S_X = \{x \in X; \ \|x\| = 1\}$ be the unit sphere of X, and $x \in S_X$. We set

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t} = \nabla^+(x, h).$$

Uniformities

The norm is strongly sub-differentiable at $x \in S_X$ if we have

$$\sup_{h \in S_X} \left(\|x + th\| - 1 - t\nabla^+(x, h) \right) = o(t)$$

when $t \to 0^+$.

We may have instead uniformity on $x \in S_X$ for a given direction $h \neq 0$. When we have

$$\sup_{x \in S_X} \left(\|x + th\| - 1 - t\nabla^+(x, h) \right) = o(t) \tag{1}$$

we say that the norm is uniformly sub-differentiable in the direction $h \in S_X$.

Uniformities

The norm is strongly sub-differentiable at $x \in S_X$ if we have

$$\sup_{h \in S_X} (\|x + th\| - 1 - t\nabla^+(x, h)) = o(t)$$

when $t \to 0^+$.

We may have instead uniformity on $x \in S_X$ for a given direction $h \neq 0$. When we have

$$\sup_{x \in S_X} (\|x + th\| - 1 - t\nabla^+(x, h)) = o(t) \tag{1}$$

we say that the norm is uniformly sub-differentiable in the direction $h \in S_X$.

More uniformities.

If $M \subset B_X$, the norm is M-uniformly sub-differentiable if

$$\sup_{(x,h)\in S_X\times M} (\|x+th\| - 1 - t\nabla^+(x,h)) = o(t) \tag{2}$$

In the special case $M = S_X$, the norm is uniformly sub-differentiable if

$$\sup_{(x,h)\in S_X\times S_X} (\|x+th\| - 1 - t\nabla^+(x,h)) = o(t) \tag{3}$$

It turns out that the equations (1), (2) and (3) can be improved, since (1) implies that the norm is differentiable in the direction h for every $x \in S_X$. We can therefore omit for free the term "sub" in the last three definitions.

More uniformities.

If $M \subset B_X$, the norm is M-uniformly sub-differentiable if

$$\sup_{(x,h)\in S_X\times M} (\|x+th\| - 1 - t\nabla^+(x,h)) = o(t) \qquad (2)$$

In the special case $M = S_X$, the norm is uniformly sub-differentiable if

$$\sup_{(x,h)\in S_X\times S_X} (\|x+th\| - 1 - t\nabla^+(x,h)) = o(t) \qquad (3)$$

It turns out that the equations (1), (2) and (3) can be improved, since (1) implies that the norm is differentiable in the direction h for every $x \in S_X$. We can therefore omit for free the term "sub" in the last three definitions.

a useful lemma

A convexity argument permits to show the following useful lemma.

Lemma 1: *If (1) holds, then the norm is differentiable in the direction h at every $x \in S_X$, and moreover*

$$\sup_{x \in S_X} \|x + th\| + \|x - th\| - 2 = o(t).$$

This lemma shows that condition (1) means that the norm is uniformly Gateaux-differentiable in the direction h. Condition (2) states that the norm is M-uniformly Gateaux-differentiable, and condition (3) is simply the uniform smoothness. Let us also note that a norm is Frechet-differentiable at a point $x \in S_X$ if and only if it is Gateaux-differentiable and strongly sub-differentiable.

a useful lemma

A convexity argument permits to show the following useful lemma.

Lemma 1: *If (1) holds, then the norm is differentiable in the direction h at every $x \in S_X$, and moreover*

$$\sup_{x \in S_X} \|x + th\| + \|x - th\| - 2 = o(t).$$

This lemma shows that condition (1) means that the norm is uniformly Gateaux-differentiable in the direction h. Condition (2) states that the norm is M-uniformly Gateaux-differentiable, and condition (3) is simply the uniform smoothness. Let us also note that a norm is Frechet-differentiable at a point $x \in S_X$ if and only if it is Gateaux-differentiable and strongly sub-differentiable.

SSD norms

The norm is strongly sub-differentiable (in short, *SSD*) if for every $x \in S_X$,

$$\sup_{h \in S_X} \left(\|x + th\| - 1 - t\nabla^+(x, h) \right) = o(t)$$

The following proposition extends a classical result on Frechet-differentiable norms.

Proposition 2: *Let X be a Banach space whose norm is SSD. Then X is an Asplund space, that is, for every separable subspace $Y \subset X$, the dual Y^* is separable.*

The proof of this proposition relies on a dual characterization of the SSD property, together with a proper use of Simons' inequality - which is a version of James' characterization of weakly compact sets in the separable case. In fact, the proof shows the following geometrical result.

SSD norms

The norm is strongly sub-differentiable (in short, *SSD*) if for every $x \in S_X$,

$$\sup_{h \in S_X} \left(\|x + th\| - 1 - t\nabla^+(x,h) \right) = o(t)$$

The following proposition extends a classical result on Frechet-differentiable norms.

Proposition 2: *Let X be a Banach space whose norm is SSD. Then X is an Asplund space, that is, for every separable subspace $Y \subset X$, the dual Y^* is separable.*

The proof of this proposition relies on a dual characterization of the SSD property, together with a proper use of Simons' inequality - which is a version of James' characterization of weakly compact sets in the separable case. In fact, the proof shows the following geometrical result.

SSD norms

The norm is strongly sub-differentiable (in short, *SSD*) if for every $x \in S_X$,

$$\sup_{h \in S_X} \left(\|x + th\| - 1 - t\nabla^+(x, h) \right) = o(t)$$

The following proposition extends a classical result on Frechet-differentiable norms.

Proposition 2: *Let X be a Banach space whose norm is SSD. Then X is an Asplund space, that is, for every separable subspace $Y \subset X$, the dual Y^* is separable.*

The proof of this proposition relies on a dual characterization of the SSD property, together with a proper use of Simons' inequality - which is a version of James' characterization of weakly compact sets in the separable case. In fact, the proof shows the following geometrical result.

The ball topology.

The ball topology b_X on a Banach space X, equipped with a given norm, is the coarsest topology for which the closed balls (for the norm) are closed. This topology is not Hausdorff on X, but sometimes it is Hausdorff on bounded subsets of X. In the case of spaces with an SSD norm, it coincides with the weak topology on bounded subsets. We can state this as follows:

Theorem 3:*Let X be a Banach space with a strongly sub-differentiable norm. Then any weakly closed bounded subset of X is an intersection of finite unions of balls.*

In particular, closed convex bounded subsets are intersections of finite union of balls. If the norm if Frechet-differentiable, closed convex bounded sets are intersections of balls (this is called Mazur's intersection property) but in general finite unions of balls are needed since for instance, any norm on a finite-dimensional space is *SSD*.

The ball topology.

The ball topology b_X on a Banach space X, equipped with a given norm, is the coarsest topology for which the closed balls (for the norm) are closed. This topology is not Hausdorff on X, but sometimes it is Hausdorff on bounded subsets of X. In the case of spaces with an SSD norm, it coincides with the weak topology on bounded subsets. We can state this as follows:

Theorem 3:*Let X be a Banach space with a strongly sub-differentiable norm. Then any weakly closed bounded subset of X is an intersection of finite unions of balls.*

In particular, closed convex bounded subsets are intersections of finite union of balls. If the norm if Frechet-differentiable, closed convex bounded sets are intersections of balls (this is called Mazur's intersection property) but in general finite unions of balls are needed since for instance, any norm on a finite-dimensional space is *SSD*.

The ball topology.

The ball topology b_X on a Banach space X, equipped with a given norm, is the coarsest topology for which the closed balls (for the norm) are closed. This topology is not Hausdorff on X, but sometimes it is Hausdorff on bounded subsets of X. In the case of spaces with an SSD norm, it coincides with the weak topology on bounded subsets. We can state this as follows:

Theorem 3:*Let X be a Banach space with a strongly sub-differentiable norm. Then any weakly closed bounded subset of X is an intersection of finite unions of balls.*

In particular, closed convex bounded subsets are intersections of finite union of balls. If the norm if Frechet-differentiable, closed convex bounded sets are intersections of balls (this is called Mazur's intersection property) but in general finite unions of balls are needed since for instance, any norm on a finite-dimensional space is *SSD*.

Dualizing *UG*

The norm is uniformly Gateaux-differentiable (UG) in the direction $h \in X \backslash \{0\}$ if one has

$$\sup_{x \in S_X} \|x + th\| + \|x - th\| - 2 = o(t).$$

This property is equivalent to a convexity condition on the dual norm.

Lemma 4: *The norm is UG-smooth in the direction h if and only if the following holds: if $(f_k) \subset B_{X^*}$ and $(g_k) \subset B_{X^*}$ are two sequences such that $lim_k \|f_k + g_k\| = 2$, then $lim_k (f_k - g_k)(h) = 0$.*

In particular, the norm is UG (that is, UG in every direction $h \neq 0$) if and only if two sequences in B_{X^*} such that $lim_k \|f_k + g_k\| = 2$ must satisfy $w^* - lim(f_k - g_k) = 0$.

Dualizing *UG*

The norm is uniformly Gateaux-differentiable (UG) in the direction $h \in X \setminus \{0\}$ if one has

$$\sup_{x \in S_X} \|x + th\| + \|x - th\| - 2 = o(t).$$

This property is equivalent to a convexity condition on the dual norm.

Lemma 4: *The norm is UG-smooth in the direction h if and only if the following holds: if $(f_k) \subset B_{X^*}$ and $(g_k) \subset B_{X^*}$ are two sequences such that $lim_k \|f_k + g_k\| = 2$, then $lim_k (f_k - g_k)(h) = 0$.*

In particular, the norm is UG (that is, UG in every direction $h \neq 0$) if and only if two sequences in B_{X^*} such that $lim_k \|f_k + g_k\| = 2$ must satisfy $w^* - lim(f_k - g_k) = 0$.

flat sets of directions

Uniform Gateaux-smoothness of a Banach space X implies the existence of natural weakly compact subsets of X. Let us denote $S_{n,p}$ the set of all $h \in X$ such that

$$sup\{|(x^* - y^*)(h)|;\ \|x^*\| \leq 1, \|y^*\| \leq 1, \|x^* + y^*\| \geq 2 - 2/n\} \leq 1/p$$

By Lemma 4, if the norm of X is UG, then $X = \cap_p \cup_n S_{n,p}$. What makes this observation useful is that the sets $S_{n,p}$ are somewhat weakly compact when p is large. More precisely, their w^*-closure in X^{**} remains close to X. This is the content of the next lemma.

flat sets of directions

Uniform Gateaux-smoothness of a Banach space X implies the existence of natural weakly compact subsets of X. Let us denote $S_{n,p}$ the set of all $h \in X$ such that

$$sup\{|(x^* - y^*)(h)|;\ \|x^*\| \leq 1, \|y^*\| \leq 1, \|x^* + y^*\| \geq 2 - 2/n\} \leq 1/p$$

By Lemma 4, if the norm of X is UG, then $X = \cap_p \cup_n S_{n,p}$. What makes this observation useful is that the sets $S_{n,p}$ are somewhat weakly compact when p is large. More precisely, their w^*-closure in X^{**} remains close to X. This is the content of the next lemma.

A lemma with the third dual.

Lemma 5: *Let* $K = \overline{S_{n,p}}^*$ *be the closure of* $S_{n,p}$ *in* X^{**}*. Then* $dist(x^{**}, X) \leq 1/p$ *for all* $x^{**} \in K$*.*

Indeed, one has $(X^{**}/X)^* = X^{\perp} \subset X^{***}$, and thus

$$dist(x^{**}, X) = sup\{h(x^{**});\ h \in X^{\perp},\ \|h\| \leq 1\}.$$

Pick $x^{**} \in K$, and $h \in X^{\perp}$ with norm 1. Let $(x^*_\alpha) \subset B_{X^*}$ converge to h in (X^{***}, w^*). We may and do assume that for all α et β,

$$\|x^*_\alpha + x^*_\beta\| > 2 - 2/n$$

and thus

$$|x^*_\alpha(x) - x^*_\beta(x)| \leq 1/p\ \forall x \in S_{n,p}. \tag{4}$$

Since $h \in X^{\perp}$, (4) implies that for all α,

$$|x^*_\alpha(x)| \leq 1/p\ \forall x \in S_{n,p}$$

et thus since $x^{**} \in K$, $x^{**}(x^*_\alpha) \leq 1/p$ for all α, hence $|h(x^{**})| \leq 1/p$.

Applications of the Lemma.

A first application is the relative weak compactness of any set M such that a norm is M-UG-smooth.

Proposition 5: *Let X be a Banach space, and $M \subset B_X$ such that*

$$\sup_{(x,h)\in S_X \times M} (\|x + th\| - 1 - t\nabla^+(x,h)) = o(t)$$

Then M is relatively weakly compact in X.

Indeed, it follows from Lemma 4 that there is $n(p) \in \mathbb{N}$ such that $M \subset \cap_p S_{n(p),p}$, and then Lemma 5 concludes the proof.

Note that a special case of Proposition 5 is that uniform smooth spaces are reflexive, sine then $M = B_X$ is weakly compact. However, this Proposition is more useful for non separable Banach spaces.

Applications of the Lemma.

A first application is the relative weak compactness of any set M such that a norm is M-UG-smooth.

Proposition 5: *Let X be a Banach space, and $M \subset B_X$ such that*

$$\sup_{(x,h)\in S_X\times M} (\|x+th\| - 1 - t\nabla^+(x,h)) = o(t)$$

Then M is relatively weakly compact in X.

Indeed, it follows from Lemma 4 that there is $n(p) \in \mathbb{N}$ such that $M \subset \cap_p S_{n(p),p}$, and then Lemma 5 concludes the proof.

Note that a special case of Proposition 5 is that uniform smooth spaces are reflexive, sine then $M = B_X$ is weakly compact. However, this Proposition is more useful for non separable Banach spaces.

Applications of the Lemma.

A first application is the relative weak compactness of any set M such that a norm is M-UG-smooth.

Proposition 5: *Let X be a Banach space, and $M \subset B_X$ such that*

$$\sup_{(x,h)\in S_X\times M} (\|x+th\| - 1 - t\nabla^+(x,h)) = o(t)$$

Then M is relatively weakly compact in X.

Indeed, it follows from Lemma 4 that there is $n(p) \in \mathbb{N}$ such that $M \subset \cap_p S_{n(p),p}$, and then Lemma 5 concludes the proof.

Note that a special case of Proposition 5 is that uniform smooth spaces are reflexive, sine then $M = B_X$ is weakly compact. However, this Proposition is more useful for non separable Banach spaces.

The structure of *UG*-smooth spaces.

Joram Lindenstrauss discovered that the existence of large weakly compact subsets in a Banach space X bears consequences on the structure of X. We can apply his ideas in this case.

Proposition 6:*Let X be a Banach space with an equivalent UG-smooth norm. Then X is a $K_{\sigma\delta}$ subset of (X^{**}, w^*).*

Indeed, it follows from Lemma 5 that if $K_{n,p}$ denotes the w^*-closure of $S_{n,p}$ in X^{**}, then

$$X = \cap_p \cup_n K_{n,p}. \tag{5}$$

The structure of *UG*-smooth spaces.

Joram Lindenstrauss discovered that the existence of large weakly compact subsets in a Banach space X bears consequences on the structure of X. We can apply his ideas in this case.

Proposition 6:*Let X be a Banach space with an equivalent UG-smooth norm. Then X is a $K_{\sigma\delta}$ subset of (X^{**}, w^*).*

Indeed, it follows from Lemma 5 that if $K_{n,p}$ denotes the w^*-closure of $S_{n,p}$ in X^{**}, then

$$X = \cap_p \cup_n K_{n,p}. \tag{5}$$

The structure of *UG*-smooth spaces.

Joram Lindenstrauss discovered that the existence of large weakly compact subsets in a Banach space X bears consequences on the structure of X. We can apply his ideas in this case.

Proposition 6:*Let X be a Banach space with an equivalent UG-smooth norm. Then X is a $K_{\sigma\delta}$ subset of (X^{**}, w^*).*

Indeed, it follows from Lemma 5 that if $K_{n,p}$ denotes the w^*-closure of $S_{n,p}$ in X^{**}, then

$$X = \cap_p \cup_n K_{n,p}. \tag{5}$$

Hilbert-generated spaces

Equation (5) is valid regardless of the density character of X, and shows that UG-smooth spaces satisfy a countable separation property, which implies in particular the existence of a long sequence of projections which breaks the space into subspaces of smaller density character. We now state a definition.

Definition 7: *A Banach space Y is Hilbert-generated is there exists a Hilbert-space $l_2(\Gamma)$ and an operator $T : l_2(\Gamma) \to Y$ with dense range.*

With this definition, we can state a characterization of UG-smooth Banach spaces, due to M. Fabian, V. Zizler and G.G.

Theorem 8:*A Banach space X has an equivalent UG-smooth norm if and only if X is isomorphic to a subspace of a Hilbert-generated Banach space.*

Hilbert-generated spaces

Equation (5) is valid regardless of the density character of X, and shows that UG-smooth spaces satisfy a countable separation property, which implies in particular the existence of a long sequence of projections which breaks the space into subspaces of smaller density character. We now state a definition.

Definition 7: *A Banach space Y is Hilbert-generated is there exists a Hilbert-space $l_2(\Gamma)$ and an operator $T : l_2(\Gamma) \to Y$ with dense range.*

With this definition, we can state a characterization of UG-smooth Banach spaces, due to M. Fabian, V. Zizler and G.G.

Theorem 8:*A Banach space X has an equivalent UG-smooth norm if and only if X is isomorphic to a subspace of a Hilbert-generated Banach space.*

Hilbert-generated spaces

Equation (5) is valid regardless of the density character of X, and shows that UG-smooth spaces satisfy a countable separation property, which implies in particular the existence of a long sequence of projections which breaks the space into subspaces of smaller density character. We now state a definition.

Definition 7: *A Banach space Y is Hilbert-generated is there exists a Hilbert-space $l_2(\Gamma)$ and an operator $T : l_2(\Gamma) \rightarrow Y$ with dense range.*

With this definition, we can state a characterization of UG-smooth Banach spaces, due to M. Fabian, V. Zizler and G.G.

Theorem 8:*A Banach space X has an equivalent UG-smooth norm if and only if X is isomorphic to a subspace of a Hilbert-generated Banach space.*

Hilbert-generated spaces

Equation (5) is valid regardless of the density character of X, and shows that UG-smooth spaces satisfy a countable separation property, which implies in particular the existence of a long sequence of projections which breaks the space into subspaces of smaller density character. We now state a definition.

Definition 7: *A Banach space Y is Hilbert-generated is there exists a Hilbert-space $l_2(\Gamma)$ and an operator $T : l_2(\Gamma) \to Y$ with dense range.*

With this definition, we can state a characterization of UG-smooth Banach spaces, due to M. Fabian, V. Zizler and G.G.

Theorem 8:*A Banach space X has an equivalent UG-smooth norm if and only if X is isomorphic to a subspace of a Hilbert-generated Banach space.*

Applications.

In the equivalence of Theorem 8, the easy direction is the existence of a UG-smooth equivalent norm on Hilbert-generated spaces Y^* (and thus on their subspaces). In the notation of Definition 7, it suffices indeed to define an equivalent dual norm on Y^* by the formula

$$N^*(y^*)^2 = \|y^*\|_*^2 + \|T^*(y^*)\|_2^2.$$

What Theorem 8 asserts in that any construction of UG-smooth equivalent norms boils down to this argument. It should be noted that UG-smoothness is the only smoothness condition which leads to a characterization (of existence of such equivalent norms on an arbitrary Banach space X).

Applications.

In the equivalence of Theorem 8, the easy direction is the existence of a *UG*-smooth equivalent norm on Hilbert-generated spaces Y^* (and thus on their subspaces). In the notation of Definition 7, it suffices indeed to define an equivalent dual norm on Y^* by the formula

$$N^*(y^*)^2 = \|y^*\|_*^2 + \|T^*(y^*)\|_2^2.$$

What Theorem 8 asserts in that any construction of *UG*-smooth equivalent norms boils down to this argument. It should be noted that *UG*-smoothness is the only smoothness condition which leads to a characterization (of existence of such equivalent norms on an arbitrary Banach space X).

Applications.

We recall that a compact space K is called uniformly Eberlein if K is homeomorphic to a weakly compact subset of a Hilbert space. In the special case of $\mathcal{C}(K)$-spaces, Theorem 8 implies the following.

Corollary 9:*Let K be a compact space. Then K is uniformly Eberlein if and only if $\mathcal{C}(K)$ has an equivalent UG-smooth norm.*

This corollary immediately implies the result by Benyamini-Rudin-Wage that the class of uniformly Eberlein compact sets is closed under continuous images.

Applications.

We recall that a compact space K is called uniformly Eberlein if K is homeomorphic to a weakly compact subset of a Hilbert space. In the special case of $\mathcal{C}(K)$-spaces, Theorem 8 implies the following.

Corollary 9:*Let K be a compact space. Then K is uniformly Eberlein if and only if $\mathcal{C}(K)$ has an equivalent UG-smooth norm.*

This corollary immediately implies the result by Benyamini-Rudin-Wage that the class of uniformly Eberlein compact sets is closed under continuous images.

Applications.

We recall that a compact space K is called uniformly Eberlein if K is homeomorphic to a weakly compact subset of a Hilbert space. In the special case of $\mathcal{C}(K)$-spaces, Theorem 8 implies the following.

Corollary 9:*Let K be a compact space. Then K is uniformly Eberlein if and only if $\mathcal{C}(K)$ has an equivalent UG-smooth norm.*

This corollary immediately implies the result by Benyamini-Rudin-Wage that the class of uniformly Eberlein compact sets is closed under continuous images.

A triibute to Joram Lindenstrauss

Let us conclude this talk by mentioning a seminal question stated by Joram Lindenstrauss when he founded the modern theory of non separable Banach spaces: is it true that a smooth Banach space is necessarily weakly compactly generated (WCG) ? Recall that a space is called WCG if it contains a weakly compact subset which spans a dense linear subspace.

In accordance with Lindenstrauss' intuition, it can be showed with Theorem 8 (and some more work) that a Banach space which has a *UG*-smooth equivalent norm and a Frechet-smooth equivalent norm is WCG.

A triibute to Joram Lindenstrauss

Let us conclude this talk by mentioning a seminal question stated by Joram Lindenstrauss when he founded the modern theory of non separable Banach spaces: is it true that a smooth Banach space is necessarily weakly compactly generated (WCG) ? Recall that a space is called WCG if it contains a weakly compact subset which spans a dense linear subspace.

In accordance with Lindenstrauss' intuition, it can be showed with Theorem 8 (and some more work) that a Banach space which has a UG-smooth equivalent norm and a Frechet-smooth equivalent norm is WCG.

THANK YOU VERY MUCH !