Decompositions and projections with respect to some classes of non-symmetric cones

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What is symmetric cone?

Definition (Symmetric cone)

A closed convex cone $\mathcal{K}$ in $\mathbb{V}$ is called a *symmetric cone* if it is self-dual, i.e.,

$$\mathcal{K} = \mathcal{K}^* := \{y \in \mathbb{V} \,|\, \langle x, y \rangle \geq 0 \quad \forall x \in \mathcal{K}\},$$

and homogeneous, i.e., for any two elements $x, y \in \mathrm{int}\mathbb{K}$ (the interior of $\mathcal{K}$), there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(\mathcal{K}) = \mathcal{K}$ and $\Gamma(x) = y$.

Theorem (Symmetric cone in Euclidean Jordan algebra)

In Euclidean Jordan algebra $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$, the symmetric cone is the set of squares, i.e., $\mathcal{K} := \{x \circ x \,|\, x \in \mathbb{V}\}$.

Definition of Jordan algebra

Definition

Let $\mathbb{V}$ be a vector space over the field of real numbers. $(\mathbb{V}, \circ)$ is called a Jordan algebra if there is a bilinear mapping $\circ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ satisfying

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 := x \circ x$.

Example (Examples of Jordan algebra)

Every associative algebra becomes a Jordan algebra under $x \circ y = \frac{1}{2}(xy + yx)$, for instance, $Sym(n, \mathbb{R})$, $C[0,1]$, $\mathcal{L}(H)$.

Definition of Euclidean Jordan algebra

Definition

A finite dimensional Jordan algebra $\mathbb{V}$ is *Euclidean* if it is formally real, that is,

$$x^2 + y^2 = 0 \quad \Longrightarrow \quad x = y = 0.$$

Equivalently, there exists an associative inner product such that

$$\langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}} \quad \forall x, y, z \in \mathbb{V}.$$

In other words, a Euclidean Jordan algebra is a triple $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle_{\mathbb{V}})$, satisfying the following three conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 := x \circ x$;

(iii) $\langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Example (Examples of symmetric cones)

- The space of real $n \times n$ symmetric matrices $\mathcal{S}^n$ under $X \circ Y = \frac{1}{2}(XY + YX)$ and $\langle X, Y \rangle_{\mathbb{V}} = \mathrm{tr}(XY)$ is a Euclidean Jordan algebra with symmetric cone $\mathcal{K} := \mathcal{S}^n_+$ which is the set of all positive semidefinite matrices.
- The space $\mathbb{R} \times \mathbb{R}^{n-1}$ under "Jordan product"

$$(x_1, x_2) \circ (y_1, y_2) = (x_1 y_1 + \langle x_2, y_2 \rangle, x_1 y_2 + y_1 x_2)$$

and $\langle (x_1, x_2), (y_1, y_2) \rangle_{\mathbb{V}} = x_1 y_1 + x_2^T y_2$ is a Euclidean Jordan algebra with symmetric cone

$$\mathbb{L}^n := \{(x_1, x_2) \mid \|x_2\| \le x_1\}$$

which is usually called second-order cone or Lorentz cone.

Symmetric cones vs Non-symmetric cones

Symmetric cones

Symmetric cones include $\mathbb{R}^n_+$, $\mathbb{L}^n$, $\mathcal{S}^n_+$ as special cases and can be unified under Euclidean Jordan algebra.

Non-symmetric cones

- Plenty of non-symmetric cones in reality.
- There is no unified framework for non-symmetric cones. How to classify them?
- Like PDEs (elliptic, hyperbolic, and parabolic), we try to classify non-symmetric cones by looking into their structures.
- Some non-symmetric cones have connection to symmetric cones. For example, circular cones, elliptic cones, and ellipsoidal cones.

Examples of non-symmetric cones (1)

Example (Some well-known non-symmetric cones)

- Circular cone:

$$\mathcal{L}_\theta := \{x \in \mathbb{R}^n \mid \|x\| \cos\theta \leq x_1\}.$$

- *p*-order cone: $(p > 1, p \neq 2)$

$$\mathcal{K}_p := \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\|_p \leq x_1\right\}.$$

- Geometric cone:

$$\mathcal{G}^n := \left\{(x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \;\middle|\; \sum_{i=1}^n e^{-\frac{x_i}{\theta}} \leq 1\right\}.$$

Examples of non-symmetric cones (2)

Example (Some well-known non-symmetric cones)

- L^p cone: (here $p = (p_1, p_2, \cdots, p_n) \in \mathbb{R}^n$ with $p_i > 1$)

$$L^p := \left\{ (x, \theta, k) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \;\middle|\; \sum_{i=1}^n \frac{|x_i|^{p_i}}{p_i \theta^{p_i - 1}} \leq k \right\}.$$

- Copositive cone:

$$\mathcal{C}^n := \left\{ A \in \mathcal{S}^n \;\middle|\; x^T A x \geq 0 \ \text{ for all } x \in \mathbb{R}^n_+ \right\}.$$

- Power cone: our focus in this talk.
- Exponential cone: our focus in this talk.

Although there exists discrepancy between symmetric cones and non-symmetric cones, there are still common concepts for both types of optimization problems.

Key Elements

- Spectral decomposition associated with cone.

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- Cone-convexity and cone-monotonicity.

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Key Elements

- Spectral decomposition associated with cone.
- Smooth and nonsmooth analysis for conic-functions.
- Cone-convexity and cone-monotonicity.
- Projection onto cones.

The circular cone

The circular cone is defined as

$$\mathcal{L}_\theta := \left\{ x \in \mathbb{R}^n \,|\, \|x\| \cos\theta \le x_1 \right\}.$$

Theorem (Zhou-Chen, JNCA, 2013)

Let $\mathcal{L}_\theta$ and $\mathbb{L}^n$ be circular cone and second-order cone, respectively. Then, we have

(a) $\mathcal{L}_\theta = A^{-1}\mathbb{L}^n$ *and* $\mathbb{L}^n = A\mathcal{L}_\theta$.

(b) $A\mathbb{L}^n = \mathcal{L}_{\frac{\pi}{2}-\theta}$ *and* $\mathcal{L}_{\frac{\pi}{2}-\theta} = A^2\mathcal{L}_\theta$.

(c) $\mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta}$ *and* $(\mathcal{L}_\theta^*)^* = \mathcal{L}_\theta$.

where $A := \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}$.

The graphs of circular cones

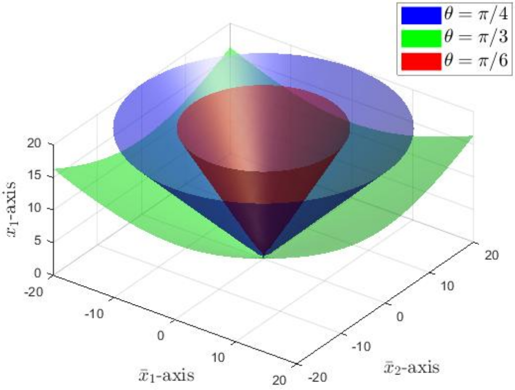


Figure: Three different circular cones in $\mathbb{R}^3$.

Spectral decomposition associated with $\mathcal{L}_\theta$

Theorem (Zhou-Chen, JNCA, 2013)

For any $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, *one has the decomposition*

$$z = \lambda_1(z) \cdot u_z^{(1)} + \lambda_2(z) \cdot u_z^{(2)}$$

where $\begin{cases} \lambda_1(z) & = & z_1 - \|z_2\| \cot\theta \\ \lambda_2(z) & = & z_1 + \|z_2\| \tan\theta \end{cases}$ *and*

$$\begin{cases} u_z^{(1)} & = & \dfrac{1}{1+\cot^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \cot\theta \cdot I \end{bmatrix} \begin{bmatrix} 1 \\ -w \end{bmatrix} = \begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)w \end{bmatrix} \\ u_z^{(2)} & = & \dfrac{1}{1+\tan^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \cdot I \end{bmatrix} \begin{bmatrix} 1 \\ w \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)w \end{bmatrix} \end{cases}$$

with $w = \dfrac{z_2}{\|z_2\|}$ *if* $z_2 \neq 0$, *and any vector in* $\mathbb{R}^{n-1}$ *satisfying* $\|w\| = 1$ *if* $z_2 = 0$.

Projection onto $\mathcal{L}_\theta$

Theorem (Zhou-Chen, JNCA, 2013)

For any $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the projection of z onto $\mathcal{L}_\theta$ is given by

$$\Pi_{\mathcal{L}_\theta}(z) = \begin{cases} z, & \text{if } z \in \mathcal{L}_\theta \\ 0, & \text{if } z \in -\mathcal{L}_\theta^* \\ u, & \text{otherwise}, \end{cases}$$

where

$$u = \begin{bmatrix} \dfrac{z_1 + \|z_2\| \tan\theta}{1 + \tan^2\theta} \\ \left(\dfrac{z_1 + \|z_2\| \tan\theta}{1 + \tan^2\theta} \tan\theta \right) \dfrac{z_2}{\|z_2\|} \end{bmatrix}.$$

The p-order cone

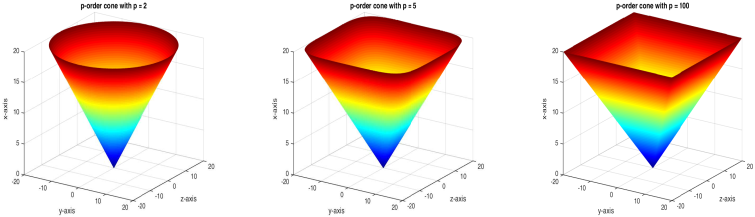
The p-order cone is defined as

$$\mathcal{K}_p := \left\{ x = (x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \|\mathbf{x_2}\|_p \le x_1 \right\}, \quad (p > 1).$$

If we write $x := (x_1, x_2, \cdots, x_n) \in \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, the p-order cone $\mathcal{K}_p$ can be equivalently expressed as

$$\mathcal{K}_p := \left\{ x \in \mathbb{R}^n \;\middle|\; x_1 \ge \left(\sum_{i=2}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\} \quad (p > 1).$$

Graphs of three different p-order cones



(a) 2-order cone (b) 5-order cone (c) 100-order cone

Figure: Three different p-order cones in $\mathbb{R}^3$.

Dual cone of p-order cone

It is well known that $\mathcal{K}_p$ is a convex cone and its dual cone is given by

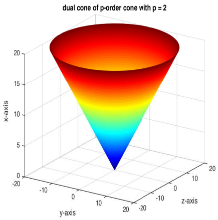
$$\mathcal{K}_p^* = \left\{ y \in \mathbb{R}^n \,\middle|\, y_1 \geq \left(\sum_{i=2}^{n} |y_i|^q \right)^{\frac{1}{q}} \right\}$$

or equivalently

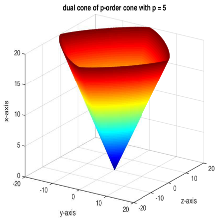
$$\mathcal{K}_p^* = \left\{ y = (y_1, \mathbf{y_2}) \in \mathbb{R} \times \mathbb{R}^{(n-1)} \mid y_1 \geq \|\mathbf{y_2}\|_q \right\} = \mathcal{K}_q,$$

where $q > 1$ and satisfies $\frac{1}{p} + \frac{1}{q} = 1$. In addition, the dual cone $\mathcal{K}_p^*$ is also a convex cone.

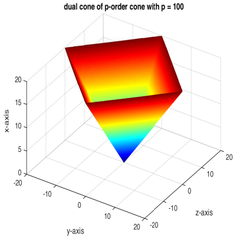
Graphs of dual cones of three different p-order cones



(a) dual of 2-order cone



(b) dual of 5-order cone



(c) dual of 100-order cone

Figure: Dual cones of three different p-order cones in $\mathbb{R}^3$.

Projection onto $\mathcal{K}_p$

Theorem (Miao-Qi-Chen, JNCA, 2017)

Let $\mathbf{z} = (z_1, \mathbf{z_2}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. *Then, the projection of* $\mathbf{z}$ *onto* $\mathcal{K}_p$ *is given by*

$$\Pi_{\mathcal{K}_p}(\mathbf{z}) = \begin{cases} \mathbf{z}, & \mathbf{z} \in \mathcal{K}_p \\ \mathbf{0}, & \mathbf{z} \in -\mathcal{K}_p^* = -\mathcal{K}_q \\ \mathbf{u}, & \textit{otherwise} \ \ (i.e., -\|\mathbf{z_2}\|_q < z_1 < \|\mathbf{z_2}\|_p) \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, \bar{\mathbf{u}})$ *with* $\bar{\mathbf{u}} = (u_2, u_3, \cdots, u_n)^T \in \mathbb{R}^{n-1}$ *satisfying*

$$u_1 = \|\bar{\mathbf{u}}\|_p = \left(|u_2|^p + |u_3|^p + \cdots + |u_n|^p\right)^{\frac{1}{p}}$$

and

$$u_i - z_i + \frac{u_1 - z_1}{u_1^{p-1}} |u_i|^{p-2} u_i = 0, \qquad \forall i = 2, \cdots, n.$$

Spectral decomposition associated with $\mathcal{K}_p$

Theorem (Miao-Qi-Chen, JNCA, 2017)

Let $\mathbf{z} = (z_1, \mathbf{z_2}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. *Then,* $\mathbf{z}$ *can be decomposed as*

$$\mathbf{z} = \alpha_1(\mathbf{z}) \cdot \mathbf{v}^{(1)}(\mathbf{z}) + \alpha_2(\mathbf{z}) \cdot \mathbf{v}^{(2)}(\mathbf{z}),$$

where

$$\left\{ \begin{array}{rcl} \alpha_1(\mathbf{z}) & = & z_1 + \|\mathbf{z_2}\|_p \\ \alpha_2(\mathbf{z}) & = & z_1 - \|\mathbf{z_2}\|_p \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} \mathbf{v}^{(1)}(\mathbf{z}) & = & \frac{1}{2} \begin{bmatrix} 1 \\ \mathbf{w_2} \end{bmatrix} \\ \mathbf{v}^{(2)}(\mathbf{z}) & = & \frac{1}{2} \begin{bmatrix} 1 \\ -\mathbf{w_2} \end{bmatrix} \end{array} \right.$$

with $\mathbf{w_2} = \frac{\mathbf{z_2}}{\|\mathbf{z_2}\|_p}$ *when* $\mathbf{z_2} \neq \mathbf{0}$*; while* $\mathbf{w_2}$ *being an arbitrary element satisfying* $\|\mathbf{w_2}\|_p = 1$ *when* $\mathbf{z_2} = \mathbf{0}$.

The geometric cone is defined as

$$\mathcal{G}^n = \left\{ (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \,\middle|\, \sum_{i=1}^{n} e^{-\frac{x_i}{\theta}} \leq 1 \right\}.$$

Note that $\mathcal{G}^n$ is solid (i.e., $\mathrm{int}\,\mathcal{G}^n \neq \emptyset$), pointed (i.e., $\mathcal{G}^n \cap -\mathcal{G}^n = 0$), closed convex cone, and its dual cone is given by

$$(\mathcal{G}^n)^* = \left\{ (y, \mu) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \,\middle|\, \mu \geq \sum_{y_i > 0} y_i \ln \frac{y_i}{\sum_{i=1}^{n} y_i} \right\}$$

where $\mu \in \mathbb{R}_+$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n_+$.

When $n = 1$, we note that the geometric cone $\mathcal{G}^1$ is just nonnegative orthant $\mathbb{R}^2_+$.

The boundary of the geometric cone $\mathcal{G}^n$ and its dual cone $(\mathcal{G}^n)^*$ can be respectively expressed as follows:

$$\mathrm{bd}\,\mathcal{G}^n = \left\{ (x,\theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \;\middle|\; \sum_{i=1}^n e^{-\frac{x_i}{\theta}} = 1 \right\}$$

and

$$\mathrm{bd}\,(\mathcal{G}^n)^* = \left\{ (y,\mu) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \;\middle|\; \mu = \sum_{y_i > 0} y_i \ln \frac{y_i}{\sum_{i=1}^n y_i} \right\}.$$

Graph of geometric cone

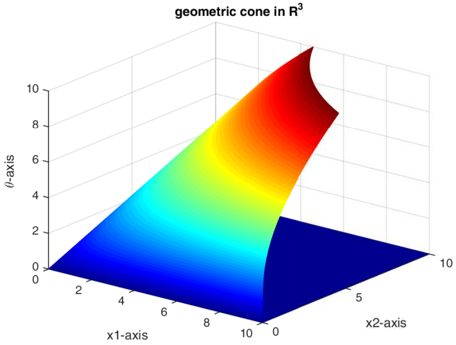


Figure: The graph of geometric cone

Graph of dual of geometric cone

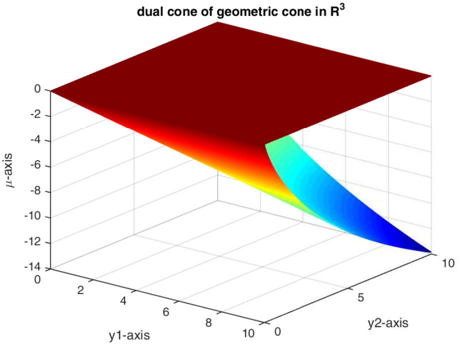


Figure: The graph of dual of geometric cone

Projection onto geometric cone

Theorem (Miao-Lu-Chen, PJO, 2018)

Let $\mathbf{x} = (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}$. *Then, the projection of* $\mathbf{x}$ *onto the geometric cone* $\mathcal{G}^n$ *is given by*

$$\Pi_{\mathcal{G}^n}(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \in \mathcal{G}^n, \\ 0, & \text{if } \mathbf{x} \in (\mathcal{G}^n)^\circ, \\ \mathbf{u}, & \text{otherwise}, \end{cases} \tag{2}$$

where $\mathbf{u} = (u, \lambda) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ *with* $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n_+$ *satisfying*

$$u_i - x_i + \frac{\lambda(\lambda - \theta)}{\sum_{i=1}^n e^{-\frac{u_i}{\lambda}} u_i} e^{-\frac{u_i}{\lambda}} = 0, \quad i = 1, 2, \cdots, n \tag{3}$$

and

$$\sum_{i=1}^n e^{-\frac{u_i}{\lambda}} = 1. \tag{4}$$

Some deficiency

- Even though we figure out the projection onto geometric cone, it is not an explicit formula because it is hard to solve equations (3)-(4).
- The decomposition associated with geometric cone is still unknown so that the corresponding nonsmooth analysis for its cone-functions is not established.

Two core non-symmetric cones

Now, we will pay attention to two very special non-symmetric cones, power cone and exponential cone.

These two cones can be viewed as core non-symmetric cones for some reasons.

The power cone

The power cone is defined by

$$\mathcal{K}_\alpha := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \;\middle|\; \bar{x}_1^{\alpha_1} \bar{x}_2^{\alpha_2} \geq |x_1|,\ \bar{x}_i \geq 0,\ i = 1, 2 \right\},$$

where $\bar{x} := (\bar{x}_1, \bar{x}_2)^T \in \mathbb{R}^2$, $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 + \alpha_2 = 1$.

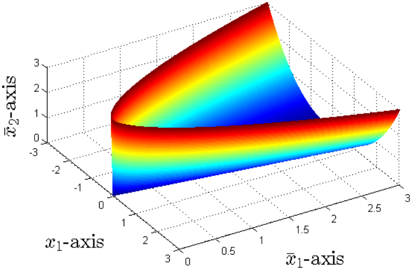


Figure: The graph of power cone $\mathcal{K}_\alpha$.

The exponential cone is defined by

$$\mathcal{K}_{\exp} := \text{cl}\left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, \bar{x}_2 \cdot \exp\left(\frac{\bar{x}_1}{\bar{x}_2}\right) \leq x_1,\ \bar{x}_2 > 0,\ x_1 \geq 0\right\},$$

where $\bar{x} := (\bar{x}_1, \bar{x}_2)^T \in \mathbb{R}^2$ and $\text{cl}(\Omega)$ denotes the closure of Ω.

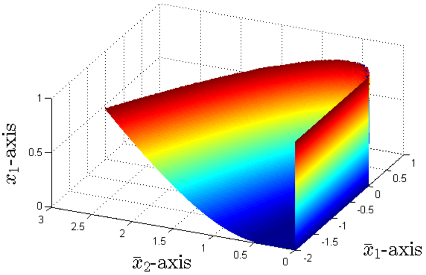


Figure: The graph of exponential cone $\mathcal{K}_{\exp}$.

Why study these two cones?

Question

Why do we pay attention to these two core non-symmetric cones (power cone $\mathcal{K}_\alpha$ and exponential cone $\mathcal{K}_{\text{exp}}$)?

Answer

- These two non-symmetric cones appear in a lot of practical applications such as location problems and geometric programming.
- Through appropriate transformations (α-representation and extended α-representation), many non-symmetric cones can be generated by the power cone $\mathcal{K}_\alpha$ and the exponential cone $\mathcal{K}_{\text{exp}}$, see Chares, Ph.D. Thesis, 2009.

(a) Second-order cone:

$$\mathbb{L}^n := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \|\bar{x}\| \le x_1 \right\},$$

where $\|\bar{x}\|$ stands for the classical Euclidean norm of a point $\bar{x} \in \mathbb{R}^{n-1}$.

(b) p-order cone:

$$\mathcal{P}_p^{(n)} := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \;\middle|\; \|\bar{x}\|_p \le x_1 \right\},$$

where $\|\bar{x}\|_p$ ($p \ge 1$) denotes the p-norm of a point $\bar{x} \in \mathbb{R}^{n-1}$, i.e.,

$$\|\bar{x}\|_p := \left(\sum_{i=1}^{n-1} |\bar{x}_i|^p \right)^{\frac{1}{p}}.$$

(c) Geometric cone:

$$\mathcal{G}^n := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \exp\left(-\frac{\bar{x}_i}{x_1} \right) \leq 1,\ x_1 \geq 0,\ \bar{x}_i \geq 0,\ i = 1, 2$$

where for $x_1 = 0$ we define $\exp\left(-\frac{\bar{x}_i}{x_1}\right) = 0$.

(d) L_p cone:

$$\mathcal{L}^p := \left\{ (x, \bar{x}) \in \mathbb{R}^2 \times \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \frac{1}{p_i} \left(\frac{|\bar{x}_i|}{x_1} \right) \leq \frac{x_2}{x_1} \right\},$$

where the parameter $p_i > 0$ for $i = 1, 2, \cdots, n$.

(e) Geometric mean's hypo-graph cone:

$$\mathcal{C}_{GM} := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, \sqrt{\bar{x}_1 \bar{x}_2} \geq x_1,\ \bar{x}_i \geq 0,\ i = 1,2 \right\}.$$

(f) Unhomogeneous power cone:

$$\mathcal{C}_\alpha := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, \bar{x}_1^{\alpha_1} \bar{x}_2^{\alpha_2} \geq |x_1|,\ \bar{x}_i \geq 0,\ i = 1,2 \right\},$$

where the parameters $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 + \alpha_2 \leq 1$.

(g) Unhomogeneous p-order cone:

$$\mathcal{C}_p := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} |\bar{x}_i|^{p_i} \leq x_1^p,\ x_1 \geq 0 \right\},$$

where $1 \leq p \leq \min_{1,2,\cdots,n} p_i$.

(h) High-dimensional power cone:

$$\mathcal{K}_\alpha^{(n)} := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n \,\middle|\, \prod_{i=1}^{n} \bar{x}_i^{\alpha_i} \geq |x_1|,\ \bar{x}_i \geq 0,\ i = 1, 2, \cdots, n \right\},$$

where the parameter $\alpha_i \in (0,1)$ and $\sum_{i=1}^{n} \alpha_i = 1$.

(i) High-dimensional p-order-power cone:

$$\mathcal{K}_{\alpha,p}^{(m,n)} := \left\{ (x,\bar{x}) \in \mathbb{R}^m \times \mathbb{R}^n \,\middle|\, \prod_{i=1}^{n} \bar{x}_i^{\alpha_i} \geq \|x\|_p, \;\; \bar{x}_i \geq 0, \;\; i = 1,2,\cdots,n \right\}$$

where the parameter $\alpha_i \in (0,1)$ and $\sum_{i=1}^{n} \alpha_i = 1$.

(j) High-dimensional power-exponential cone:

$$\begin{aligned} & \mathcal{K}_{\exp,\alpha} \\ := \; & \left\{ (x,\bar{x}) \in \mathbb{R}^2 \times \mathbb{R}^n \,\middle|\, x_2 \cdot \exp\left(\frac{x_1}{x_2}\right) \leq \prod_{i=1}^{n} \bar{x}_i^{\alpha_i}, \;\; x_2 > 0, \;\; \bar{x}_i > 0, \;\; i = 1,2,\cdots,n \right\} \end{aligned}$$

There exists a generalized power cone, which is defined as

$$\mathcal{K}^{\alpha}_{m,n} := \left\{ (x,z) \in \mathbb{R}^m_+ \times \mathbb{R}^n \;\middle|\; \|z\| \leq \prod_{i=1}^{m} x_i^{\alpha_i} \right\}$$

where $\alpha_i > 0$ and $\sum_{i=1}^m \alpha_i = 1$, $x = (x_1, \cdots, x_m) \in \mathbb{R}^m_+$, $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$.

Indeed, its dual cone is given by

$$(\mathcal{K}^{\alpha}_{m,n})^* = \left\{ (\lambda, y) \in \mathbb{R}^m_+ \times \mathbb{R}^n \;\middle|\; \|y\| \leq \prod_{i=1}^{m} \left(\frac{\lambda_i}{\alpha_i} \right)^{\alpha_i} \right\}$$

where $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m_+$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

Graphs of generalized power cones (1)

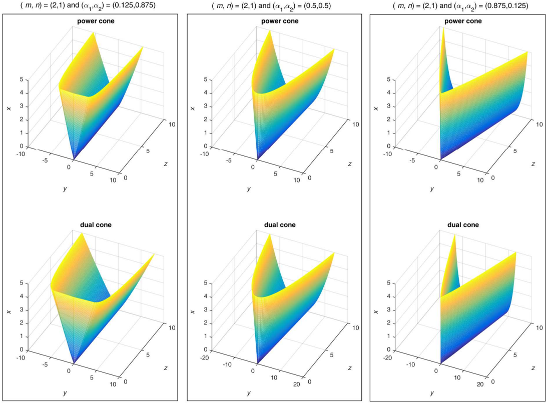


Figure: The 3-dimensional power cones and its dual cones with $m = 2, n = 1$ and different α_1, α_2

Graphs of generalized power cone (2)

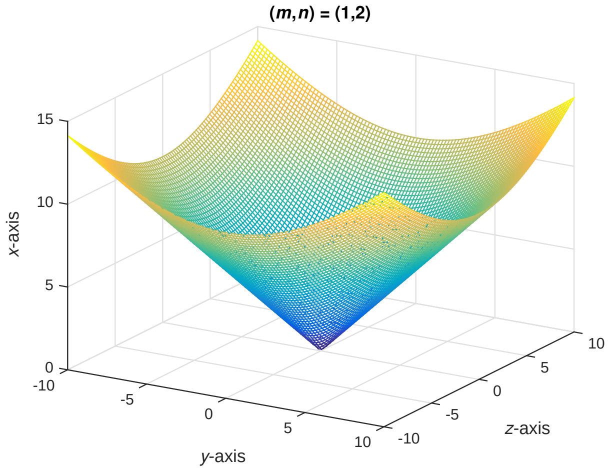


Figure: The 3-dimensional power cone with $m = 1, n = 2$, i.e., second-order cone

Projection onto generalized power cone (1)

Theorem (Hien, MMOR, 2015)

Let $(x,z) \in \mathbb{R}^m \times \mathbb{R}^n$ *with* $x = (x_1, \cdots, x_m)^T \in \mathbb{R}^m$ *and* $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$. *Set* $(\hat{x}, \hat{z})$ *be the projection of* (x,z) *onto the generalized power cone* $\mathcal{K}^{\alpha}_{m,n}$. *Denote*

$$\Phi(x,z,r) = \frac{1}{2}\prod_{i=1}^{m}\left(x_i + \sqrt{x_i^2 + 4\alpha_i r(\|z\| - r)}\right)^{\alpha_i} - r.$$

(a) *If* $(x,z) \notin \mathcal{K}^{\alpha}_{m,n} \cup -(\mathcal{K}^{\alpha}_{m,n})^*$ *and* $z \neq 0$, *then its projection onto* $\mathcal{K}^{\alpha}_{m,n}$ *is*

$$\begin{cases} \hat{x}_i = \frac{1}{2}\left(x_i + \sqrt{x_i^2 + 4\alpha_i r(\|z\| - r)}\right), & i = 1, \cdots, m, \\ \hat{z}_l = z_l \frac{r}{\|z\|}, & l = 1, \cdots, n, \end{cases}$$

where $r = r(x,z)$ *is the unique solution to the system:*

Projection onto generalized power cone (2)

Theorem (Hien, MMOR, 2015)

$$E(x,z): \quad \begin{cases} \Phi(x,z,r) = 0, \\ 0 < r < \|z\|. \end{cases}$$

(b) *If $(x,z) \notin \mathcal{K}^{\alpha}_{m,n} \cup -(\mathcal{K}^{\alpha}_{m,n})^*$ and $z = 0$, then its projection onto $\mathcal{K}^{\alpha}_{m,n}$ is*

$$\begin{cases} \hat{x}_i = (x_i)_+ = \max\{0, x_i\}, \quad i = 1, \cdots, m, \\ \hat{z}_l = 0, \quad l = 1, \cdots, n. \end{cases}$$

(c) *If $(x,z) \in \mathcal{K}^{\alpha}_{m,n}$, then its projection onto $\mathcal{K}^{\alpha}_{m,n}$ is itself, i.e., $(\hat{x},\hat{z}) = (x,z)$.*

(d) *If $(x,z) \in -(\mathcal{K}^{\alpha}_{m,n})^*$, then its projection onto $\mathcal{K}^{\alpha}_{m,n}$ is zero vector, i.e., $(\hat{x},\hat{z}) = 0$.*

Projection onto exponential cone

Theorem (Miao-Lu-Chen, PJO, 2018)

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then the projection of x onto the exponential cone $\mathcal{K}_e$ is given by

$$\Pi_{\mathcal{K}_e}(x) = \begin{cases} x, & \text{if } x \in \mathcal{K}_e, \\ 0, & \text{if } x \in (\mathcal{K}_e)^\circ, \\ v, & \text{otherwise,} \end{cases} \tag{5}$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ has the following form:

(a) *if $x_1 \leq 0$ and $x_2 \leq 0$, then $v = (x_1, 0, \frac{x_3 + |x_3|}{2})$.*

(b) *otherwise, the projection $\Pi_{\mathcal{K}_e}(x) = v$ satisfies the equations:*

$$\begin{aligned} v_1 - x_1 + e^{\frac{v_1}{v_2}}\left(v_2 e^{\frac{v_1}{v_2}} - x_3\right) &= 0, \\ v_2(v_2 - x_2) - (v_1 - x_1)(v_2 - v_1) &= 0, \\ v_2 e^{\frac{v_1}{v_2}} &= v_3. \end{aligned}$$

Still not good enough

Remarks

- Even though the projections onto power cone and exponential cone are available, there are not in explicit expressions because there (respectively) needs to solve some system to achieve the expression, which is hard.
- Looking for explicit expressions for the projections onto power cone and exponential cone are still desirable.

Question

Is it possible to achieve the explicit decomposition expressions of the power cone $\mathcal{K}_\alpha$ and the exponential cone $\mathcal{K}_{\exp}$?

Answer

Two types of decompositions will be provided.

- One motivation is based on observations from Moreau decomposition.
- The other motivation comes from geometric structures of these two core cones.

Moreau Decomposition Theorem

Theorem (Moreau Decomposition Theorem)

Let $\mathcal{K}$ be a closed convex cone. For any given $z \in \mathbb{R}^n$, one can decompose z as follows:

$$z = \Pi_{\mathcal{K}}(z) + \Pi_{\mathcal{K}^\circ}(z) = \Pi_{\mathcal{K}}(z) - \Pi_{\mathcal{K}^*}(-z),$$

where $\Pi_{\mathcal{K}}(z)$ stands for the metric projection of $z \in \mathbb{R}^n$ onto $\mathcal{K}$, while $\Pi_{\mathcal{K}^\circ}(z)$ means the projection of z onto the polar cone $\mathcal{K}^\circ$.

Remark: The polar cone is defined by $\mathcal{K}^\circ := \{y \in \mathbb{R}^n \,|\, x^T y \leq 0, \ \forall x \in \mathcal{K}\}$. Traditionally, we use $\mathcal{K}^*$ to denote the dual cone of $\mathcal{K}$, where $\mathcal{K}^* = -\mathcal{K}^\circ$.

About the Moreau Decomposition

- For some famous symmetric cones, like SOC and $\mathcal{S}^n_+$, we can define the corresponding conic functions such as SOC-function and Löwner's operator. Accordingly, one can further establish their analytic properties and design numerical algorithms based on this Moreau decomposition.
- However, due to the lack of explicit expressions for projections onto non-symmetric cones, one cannot employ this classical theorem directly to non-symmetric cones.

The second-order cone (SOC) is defined by

$$\mathbb{L}^n := \{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|\bar{x}\| \le x_1\}.$$

For any given $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we have

$$\Pi_{\mathbb{L}^n}(z) = \max(0, \lambda_1(z)) \cdot u_z^{(1)} + \max(0, \lambda_2(z)) \cdot u_z^{(2)},$$

where

$$\lambda_i(z) := z_1 + (-1)^i \|\bar{z}\|, \quad u_z^{(i)} := \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{\bar{z}}{\|\bar{z}\|}\right) & \text{if } \bar{z} \neq 0, \\ \frac{1}{2}\left(1, (-1)^i w\right) & \text{if } \bar{z} = 0 \end{cases}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$.

Moreau Decomposition in SOC setting (2)

Moreau Decomposition w.r.t. SOC

The decomposition at $z \in \mathbb{R}^n$ with respect to $\mathbb{L}^n$ becomes

$$z = x + y, \quad x \in \mathbb{L}^n, \; y \in (\mathbb{L}^n)^\circ$$

with

$$\begin{aligned} x &:= \max(0, \lambda_1(z)) \cdot u_z^{(1)} + \max(0, \lambda_2(z)) \cdot u_z^{(2)} = \Pi_{\mathbb{L}^n}(z), \\ y &:= \min(0, \lambda_1(z)) \cdot u_z^{(1)} + \min(0, \lambda_2(z)) \cdot u_z^{(2)} = \Pi_{(\mathbb{L}^n)^\circ}(z). \end{aligned}$$

Moreau Decomposition in circular cone setting (1)

The circular cone $\mathcal{L}_\theta$ is defined by

$$\mathcal{L}_\theta := \{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|\bar{x}\| \le x_1 \tan\theta\}.$$

For any given $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we have

$$\Pi_{\mathcal{L}_\theta}(z) = \max(0, \tilde{\lambda}_1(z)) \cdot \tilde{u}_z^{(1)} + \max(0, \tilde{\lambda}_2(z)) \cdot \tilde{u}_z^{(2)},$$

where

$$\tilde{\lambda}_1(z) := z_1 - \|\bar{z}\| \cot\theta, \;\; \tilde{\lambda}_2(z) := z_1 + \|\bar{z}\| \tan\theta,$$

$$\tilde{u}_z^{(1)} := \frac{1}{1+\cot^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \cot\theta \end{bmatrix} \begin{bmatrix} 1 \\ -w \end{bmatrix},$$

$$\tilde{u}_z^{(2)} := \frac{1}{1+\tan^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix} \begin{bmatrix} 1 \\ w \end{bmatrix}$$

with $w = \frac{\bar{z}}{\|\bar{z}\|}$ if $\bar{x} \neq 0$, and any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $\bar{x} = 0$.

Moreau Decomposition in circular cone setting (2)

Moreau Decomposition w.r.t. Circular Cone

The decomposition at $z \in \mathbb{R}^n$ with respect to $\mathcal{L}_\theta$ becomes

$$z = \tilde{x} + \tilde{y}, \quad \tilde{x} \in \mathcal{L}_\theta, \ \tilde{y} \in \mathcal{L}_\theta^\circ,$$

where

$$\begin{aligned} \tilde{x} &:= \max(0, \tilde{\lambda}_1(z)) \cdot \tilde{u}_z^{(1)} + \max(0, \tilde{\lambda}_2(z)) \cdot \tilde{u}_z^{(2)} = \Pi_{\mathcal{L}_\theta}(z), \\ \tilde{y} &:= \min(0, \tilde{\lambda}_1(z)) \cdot \tilde{u}_z^{(1)} + \min(0, \tilde{\lambda}_2(z)) \cdot \tilde{u}_z^{(2)} = \Pi_{\mathcal{L}_\theta^\circ}(z). \end{aligned}$$

The main source of difficulties

Difficulties

- For most non-symmetric convex cones, computing the metric projection at any given point is also a difficult task.
- For the given point lying outside the union of the cone and its polar, the projection mapping does not have explicit formula.

To conquer these difficulties, we look into the structures of the power cone $\mathcal{K}_\alpha$ and the exponential cone $\mathcal{K}_{\exp}$.

Type I decomposition of power cone $\mathcal{K}_\alpha$: Main idea

Recall the Moreau decomposition:

$$z = \Pi_{\mathcal{K}}(z) + \Pi_{\mathcal{K}^\circ}(z).$$

As mentioned earlier, if $z \notin \mathcal{K} \cup \mathcal{K}^\circ$, we usually do not have an exact projection formulas of $\Pi_{\mathcal{K}}(z)$ and $\Pi_{\mathcal{K}^\circ}(z)$.

Thus, we wish to find two scalars $s_x, s_y \in \mathbb{R}$ and two vectors $x, y \in \mathbb{R}^n$ such that the point z can be decomposed into the form of

$$z = s_x \cdot x + s_y \cdot y, \quad x \in \partial\mathcal{K},\ y \in \partial\mathcal{K}^\circ,\ (s_x, s_y) \neq (0,0). \tag{6}$$

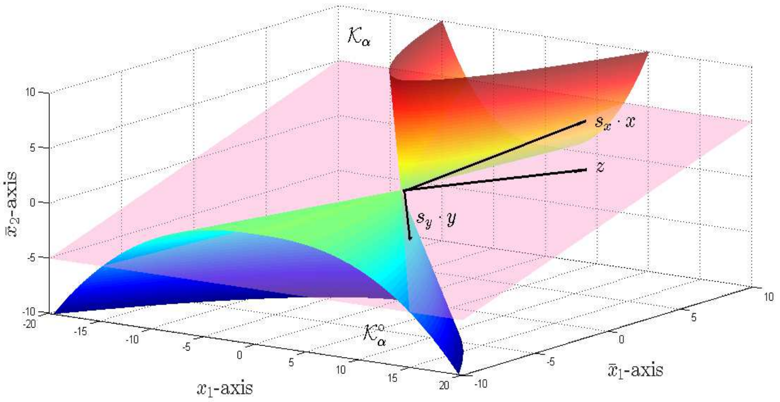


Figure: Type I decomposition of the power cone.

Type II decomposition of power cone $\mathcal{K}_\alpha$: Main idea

In view of the existing decomposition for SOC setting, we observe that any given point z can be decomposed as

$$z = s_x \cdot x + s_y \cdot y, \quad x \in \partial\mathcal{K},\ y \in \partial\mathcal{K},\ (s_x, s_y) \neq (0,0), \qquad (7)$$

where $s_x, s_y \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.

Only the boundary of the given cone is involved in formula (7) compared with Type I decomposition.

Graph for Type II decomposition of $\mathcal{K}_\alpha$

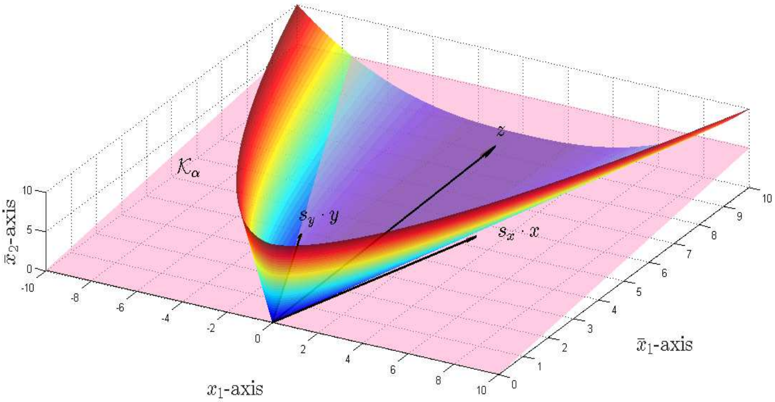


Figure: Type II decomposition of the power cone.

The dual of power cone $\mathcal{K}_\alpha$

The dual of the power cone $\mathcal{K}_\alpha$ (denoted by $\mathcal{K}_\alpha^*$) is described in the form of

$$\mathcal{K}_\alpha^* := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \;\middle|\; \left(\frac{\bar{x}_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{\bar{x}_2}{\alpha_2}\right)^{\alpha_2} \geq |x_1|, \;\; \bar{x}_i \geq 0, \;\; i = 1, 2 \right\},$$

where $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 + \alpha_2 = 1$.

Denote $\sigma_\alpha : \mathbb{R}^2_+ \to \mathbb{R}$ and $\eta_\alpha : \mathbb{R}^2_+ \to \mathbb{R}$ as follows:

$$\sigma_\alpha(\bar{x}) := \bar{x}_1^{\alpha_1} \bar{x}_2^{\alpha_2}, \quad \eta_\alpha(\bar{x}) := \left(\frac{\bar{x}_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{\bar{x}_2}{\alpha_2}\right)^{\alpha_2}.$$

Then, the boundary of $\mathcal{K}_\alpha$ and $\mathcal{K}_\alpha^*$ (denoted by $\partial\mathcal{K}_\alpha$ and $\partial\mathcal{K}_\alpha^*$) are respectively given by

$$\partial\mathcal{K}_\alpha := S_1 \cup S_2 \cup S_3 \cup \{0\}, \quad \partial\mathcal{K}_\alpha^* := S_1 \cup S_2 \cup S_4 \cup \{0\},$$

where the sets S_i $(i = 1, 2, 3, 4)$ are defined by

$$\begin{aligned}
S_1 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 > 0,\ \bar{x}_2 = 0\right\}, \\
S_2 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 = 0,\ \bar{x}_2 > 0\right\}, \\
S_3 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, |x_1| = \sigma_\alpha(\bar{x}),\ \bar{x}_1 > 0,\ \bar{x}_2 > 0\right\}, \\
S_4 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, |x_1| = \eta_\alpha(\bar{x}),\ \bar{x}_1 > 0,\ \bar{x}_2 > 0\right\}.
\end{aligned}$$

The polar of power cone $\mathcal{K}_\alpha$

The polar of $\mathcal{K}_\alpha$ (denoted by $\mathcal{K}_\alpha^\circ$) is characterized as

$$\mathcal{K}_\alpha^\circ := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, \left(\frac{-\bar{x}_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{-\bar{x}_2}{\alpha_2}\right)^{\alpha_2} \geq |x_1|,\ \bar{x}_i \leq 0,\ i = 1, 2 \right\}$$

and its boundary is given by

$$\partial \mathcal{K}_\alpha^\circ := T_1 \cup T_2 \cup T_3 \cup \{0\},$$

where the set T_i $(i = 1, 2, 3)$ are described as follows:

$$\begin{aligned}
T_1 &:= \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 < 0,\ \bar{x}_2 = 0 \right\}, \\
T_2 &:= \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 = 0,\ \bar{x}_2 < 0 \right\}, \\
T_3 &:= \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, |x_1| = \eta_\alpha(-\bar{x}),\ \bar{x}_1 < 0,\ \bar{x}_2 < 0 \right\}.
\end{aligned}$$

The set $\mathcal{K}_\alpha \cup \mathcal{K}_\alpha^\circ$ can be divided into seven parts:

$$\mathcal{K}_\alpha \cup \mathcal{K}_\alpha^\circ = S_1 \cup S_2 \cup T_1 \cup T_2 \cup P_1 \cup P_2 \cup \{0\},$$

with the sets P_1 and P_2 given by

$$\begin{aligned} P_1 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, |x_1| \leq \sigma_\alpha(\bar{x}),\ \bar{x}_1 > 0,\ \bar{x}_2 > 0\right\}, \\ P_2 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, |x_1| \leq \eta_\alpha(-\bar{x}),\ \bar{x}_1 < 0,\ \bar{x}_2 < 0\right\}. \end{aligned}$$

Notational Simplifications

The key to deriving Type I and Type II decompositions is dividing the space $\mathbb{R} \times \mathbb{R}^2$ into four blocks.

To this end, we adapt some notations that will be used in the sequel. More specifically, we let

$$\begin{aligned} z &:= (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2, \\ \bar{z} &:= (\bar{z}_1, \bar{z}_2)^T \in \mathbb{R}^2, \\ \bar{z}_{\min} &:= \min\{\bar{z}_1, \bar{z}_2\}, \\ \bar{z}_{\max} &:= \max\{\bar{z}_1, \bar{z}_2\}, \\ I_- &:= \{i \in \{1,2\} \mid \bar{z}_i < 0\}, \\ I_0 &:= \{i \in \{1,2\} \mid \bar{z}_i = 0\}, \\ I_+ &:= \{i \in \{1,2\} \mid \bar{z}_i > 0\}. \end{aligned}$$

Four blocks (1)

In light of these notations, we divide the space $\mathbb{R} \times \mathbb{R}^2$ into the following four blocks:

Block I:

$$B_1 := \left\{(z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_{\min} \cdot \bar{z}_{\max} > 0 \text{ or } (\bar{z}_{\min} = \bar{z}_{\max} = 0 \text{ and } z_1 \neq 0)\right.$$

The set B_1 includes: (i) all elements of $\bar{z}$-part is greater than 0 or less than 0. (ii) $\bar{z} = 0$ but $z_1 \neq 0$.

Block II:

$$B_2 := \left\{(z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_{\min} \cdot \bar{z}_{\max} = 0 \text{ and } \bar{z}_{\min} + \bar{z}_{\max} \neq 0\right\}.$$

The set B_2 consists of the points that all elements of $\bar{z}$-part is not greater than 0 or not less than 0 and there exist at least one zero element.

Four blocks (2)

Block III:

$$B_3 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_{\min} \cdot \bar{z}_{\max} < 0 \right\}.$$

The set B_3 contains the points that the $\bar{z}$-part has at least one element greater than 0 and at least one element less than 0.

Block IV:

$$B_4 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_{\min} = \bar{z}_{\max} = 0 \text{ and } z_1 = 0 \right\}.$$

This set includes only one point $(0, 0) \in \mathbb{R} \times \mathbb{R}^2$.

Type I decomposition of power cone (1)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

For any given $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2$, the Type I decomposition of the power cone $\mathcal{K}_\alpha$ can be described as follows:

(a) *If $z \in B_1$, then*

$$z = \begin{cases} \dot{s}_x^{(B_1,a)} \cdot \dot{x}^{(B_1,a)} + \dot{s}_y^{(B_1,a)} \cdot \dot{y}^{(B_1,a)}, & \text{if} \quad |I_-| = |I_0| = 0, \\ \dot{s}_x^{(B_1,b)} \cdot \dot{x}^{(B_1,b)} + \dot{s}_y^{(B_1,b)} \cdot \dot{y}^{(B_1,b)}, & \text{if} \quad |I_0| = |I_+| = 0, \\ \dot{s}_x^{(B_1,c)} \cdot \dot{x}^{(B_1,c)} + \dot{s}_y^{(B_1,c)} \cdot \dot{y}^{(B_1,c)}, & \text{if} \quad |I_-| = |I_+| = 0, \end{cases}$$

where $\dot{x}^{(B_1,a)}$, $\dot{y}^{(B_1,a)}$, $\dot{s}_x^{(B_1,a)}$, $\dot{s}_y^{(B_1,a)}$ are described as in (i), $\dot{x}^{(B_1,b)}$, $\dot{y}^{(B_1,b)}$, $\dot{s}_x^{(B_1,b)}$, $\dot{s}_y^{(B_1,b)}$ are described as in (ii), and $\dot{x}^{(B_1,c)}$, $\dot{y}^{(B_1,c)}$, $\dot{s}_x^{(B_1,c)}$, $\dot{s}_y^{(B_1,c)}$ are described as in (iii).

Type I decomposition of power cone (2)

(i)

$$\dot{x}^{(B_1,a)} := \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_\alpha(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_\alpha,$$

$$\dot{y}^{(B_1,a)} := \begin{bmatrix} 1 \\ -\frac{\bar{z}}{\eta_\alpha(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_\alpha^\circ,$$

$$\dot{s}_x^{(B_1,a)} := \frac{z_1 + \eta_\alpha(\bar{z})}{\sigma_\alpha(\bar{z}) + \eta_\alpha(\bar{z})} \cdot \sigma_\alpha(\bar{z}),$$

$$\dot{s}_y^{(B_1,a)} := \frac{z_1 - \sigma_\alpha(\bar{z})}{\sigma_\alpha(\bar{z}) + \eta_\alpha(\bar{z})} \cdot \eta_\alpha(\bar{z}).$$

Type I decomposition of power cone (3)

(ii)

$$\begin{aligned}
\dot{x}^{(B_1,b)} &:= \begin{bmatrix} 1 \\ \frac{-\bar{z}}{\sigma_\alpha(-\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_\alpha, \\
\dot{y}^{(B_1,b)} &:= \begin{bmatrix} 1 \\ \frac{\bar{z}}{\eta_\alpha(-\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_\alpha^\circ, \\
\dot{s}_x^{(B_1,b)} &:= \frac{z_1 - \eta_\alpha(-\bar{z})}{\sigma_\alpha(-\bar{z}) + \eta_\alpha(-\bar{z})} \cdot \sigma_\alpha(-\bar{z}), \\
\dot{s}_y^{(B_1,b)} &:= \frac{z_1 + \sigma_\alpha(-\bar{z})}{\sigma_\alpha(-\bar{z}) + \eta_\alpha(-\bar{z})} \cdot \eta_\alpha(-\bar{z}).
\end{aligned}$$

(iii) Denote $\mathbf{1} := (1,1)^T \in \mathbb{R}^2$.

$$
\begin{aligned}
\dot{x}^{(B_1,c)} &:= \begin{bmatrix} 1 \\ \frac{\mathbf{1}}{\sigma_\alpha(\mathbf{1})} \end{bmatrix} \in \partial\mathcal{K}_\alpha, \\
\dot{y}^{(B_1,c)} &:= \begin{bmatrix} 1 \\ -\frac{\mathbf{1}}{\eta_\alpha(\mathbf{1})} \end{bmatrix} \in \partial\mathcal{K}_\alpha^\circ, \\
\dot{s}_x^{(B_1,c)} &:= \frac{z_1}{\sigma_\alpha(\mathbf{1}) + \eta_\alpha(\mathbf{1})} \cdot \sigma_\alpha(\mathbf{1}), \\
\dot{s}_y^{(B_1,c)} &:= \frac{z_1}{\sigma_\alpha(\mathbf{1}) + \eta_\alpha(\mathbf{1})} \cdot \eta_\alpha(\mathbf{1}).
\end{aligned}
$$

Type I decomposition of power cone (5)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(b) *If* $z \in B_2$, *then*

$$z = \begin{cases} \dot{x}^{(B_2,a)} + \dot{y}^{(B_2,a)}, & \text{if} \quad |I_-| = 0, \\ (-1) \cdot \dot{x}^{(B_2,b)} + (-1) \cdot \dot{y}^{(B_2,b)}, & \text{if} \quad |I_+| = 0, \end{cases}$$

where $\dot{x}^{(B_2,a)}$, $\dot{y}^{(B_2,a)}$ *are described as in (i), and* $\dot{x}^{(B_2,b)}$, $\dot{y}^{(B_2,b)}$ *are described as in (ii).*

Type I decomposition of power cone (6)

(i) Let $\dot{x}^{(B_2,a)} = (\dot{x}_1^{(B_2,a)}, \dot{\bar{x}}^{(B_2,a)})$ and $\dot{y}^{(B_2,a)} = (\dot{y}_1^{(B_2,a)}, \dot{\bar{y}}^{(B_2,a)})$.

$$\begin{aligned}
\dot{x}_1^{(B_2,a)} &:= z_1, \\
\dot{\bar{x}}_j^{(B_2,a)} &:= \begin{cases} \bar{z}_j & \text{if} \quad j \in I_+, \\ 1 & \text{if} \quad j \in I_0 \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\dot{\bar{x}}_i^{(B_2,a)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if} \quad j = k. \end{cases} \\
\dot{y}_1^{(B_2,a)} &:= 0, \\
\dot{\bar{y}}_j^{(B_2,a)} &:= \begin{cases} 0 & \text{if} \quad j \in I_+, \\ -1 & \text{if} \quad j \in I_0 \text{ and } j \neq k, \\ -\left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\dot{\bar{x}}_i^{(B_2,a)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if} \quad j = k. \end{cases}
\end{aligned}$$

(ii) Let $\dot{x}^{(B_2,b)} = (\dot{x}_1^{(B_2,b)}, \dot{\bar{x}}^{(B_2,b)})$ and $\dot{y}^{(B_2,b)} = (\dot{y}_1^{(B_2,b)}, \dot{\bar{y}}^{(B_2,b)})$.

$$\dot{x}_1^{(B_2,b)} := -z_1,$$

$$\dot{\bar{x}}_j^{(B_2,b)} := \begin{cases} -\bar{z}_j & \text{if } \; j \in I_-, \\ 1 & \text{if } \; j \in I_0 \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\dot{\bar{x}}_i^{(B_2,b)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } \; j = k. \end{cases}$$

$$\dot{y}_1^{(B_2,b)} := 0,$$

$$\dot{\bar{y}}_j^{(B_2,b)} := \begin{cases} 0 & \text{if } \; j \in I_-, \\ -1 & \text{if } \; j \in I_0 \text{ and } j \neq k, \\ -\left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\dot{\bar{x}}_i^{(B_2,b)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } \; j = k. \end{cases}$$

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(c) *If* $z \in B_3$*, then* $z = \dot{x}^{(B_3)} + \dot{y}^{(B_3)}$*, where* $\dot{x}^{(B_3)} = (\dot{x}_1^{(B_3)}, \dot{\bar{x}}^{(B_3)})$ *and* $\dot{y}^{(B_3)} = (\dot{y}_1^{(B_3)}, \dot{\bar{y}}^{(B_3)})$ *are given by*

$$
\begin{aligned}
\dot{x}_1^{(B_3)} &:= z_1, \\
\dot{\bar{x}}_j^{(B_3)} &:= \begin{cases} \bar{z}_j & \text{if } j \in I_+, \\ -\bar{z}_j & \text{if } j \in I_- \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\dot{\bar{x}}_i^{(B_3)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } j = k. \end{cases} \\
\dot{y}_1^{(B_3)} &:= 0, \\
\dot{\bar{y}}_j^{(B_3)} &:= \begin{cases} 0 & \text{if } j \in I_+, \\ 2\bar{z}_j & \text{if } j \in I_- \text{ and } j \neq k, \\ \bar{z}_k - \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\dot{\bar{x}}_i^{(B_3)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } j = k. \end{cases}
\end{aligned}
$$

Type I decomposition of power cone (9)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(d) *If* $z \in B_4$, *then* $z = \dot{x}^{(B_4)} + \dot{y}^{(B_4)}$, *where* $\dot{x}^{(B_4)} = (\dot{x}_1^{(B_4)}, \dot{\bar{x}}^{(B_4)})$ *and* $\dot{y}^{(B_4)} = (\dot{y}_1^{(B_4)}, \dot{\bar{y}}^{(B_4)})$ *are given by*

$$
\begin{aligned}
\dot{x}_1^{(B_4)} &:= 0, \\
\dot{\bar{x}}^{(B_4)} &:= \mathbf{1} - \mathbf{1}_k, \\
\dot{y}_1^{(B_4)} &:= 0, \\
\dot{\bar{y}}^{(B_4)} &:= \mathbf{1}_k - \mathbf{1},
\end{aligned}
$$

with $\mathbf{1}_k$ *being the* k*th column of the identity matrix* $I_2 \in \mathbb{R}^{2\times 2}$ $(k = 1, 2)$.

Type II decomposition of power cone (1)

Using the same blocks defined as earlier and applying the 2nd idea.

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

For any given $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2$, the Type II decomposition of the power cone $\mathcal{K}_\alpha$ is described as follows:

(a) *If $z \in B_1$, then*

$$z = \begin{cases} \ddot{s}_x^{(B_1,a)} \cdot \ddot{x}^{(B_1,a)} + \ddot{s}_y^{(B_1,a)} \cdot \ddot{y}^{(B_1,a)}, & \text{if } \; |I_-| = |I_0| = 0, \\ \ddot{s}_x^{(B_1,b)} \cdot \ddot{x}^{(B_1,b)} + \ddot{s}_y^{(B_1,b)} \cdot \ddot{y}^{(B_1,b)}, & \text{if } \; |I_0| = |I_+| = 0, \\ \ddot{s}_x^{(B_1,c)} \cdot \ddot{x}^{(B_1,c)} + \ddot{s}_y^{(B_1,c)} \cdot \ddot{y}^{(B_1,c)}, & \text{if } \; |I_-| = |I_+| = 0, \end{cases}$$

where $\ddot{x}^{(B_1,a)}$, $\ddot{y}^{(B_1,a)}$, $\ddot{s}_x^{(B_1,a)}$, $\ddot{s}_y^{(B_1,a)}$ are given as in (i), $\ddot{x}^{(B_1,b)}$, $\ddot{y}^{(B_1,b)}$, $\ddot{s}_x^{(B_1,b)}$, $\ddot{s}_y^{(B_1,b)}$ are given as in (ii), and $\ddot{x}^{(B_1,c)}$, $\ddot{y}^{(B_1,c)}$, $\ddot{s}_x^{(B_1,c)}$, $\ddot{s}_y^{(B_1,c)}$ are given as in (iii).

(i)

$$
\begin{aligned}
\ddot{x}^{(B_1,a)} &:= \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_\alpha(\bar{z})} \end{bmatrix} \in \partial \mathcal{K}_\alpha, \\
\ddot{y}^{(B_1,a)} &:= \begin{bmatrix} -1 \\ \frac{\bar{z}}{\sigma_\alpha(\bar{z})} \end{bmatrix} \in \partial \mathcal{K}_\alpha, \\
\ddot{s}_x^{(B_1,a)} &:= \frac{z_1 + \sigma_\alpha(\bar{z})}{2}, \\
\ddot{s}_y^{(B_1,a)} &:= \frac{\sigma_\alpha(\bar{z}) - z_1}{2}.
\end{aligned}
$$

Type II decomposition of power cone (3)

(ii)

$$
\begin{aligned}
\ddot{x}^{(B_1,b)} &:= \begin{bmatrix} 1 \\ \frac{-\bar{z}}{\sigma_\alpha(-\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_\alpha, \\
\ddot{y}^{(B_1,b)} &:= \begin{bmatrix} -1 \\ \frac{-\bar{z}}{\sigma_\alpha(-\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_\alpha, \\
\ddot{s}_x^{(B_1,b)} &:= \frac{z_1 - \sigma_\alpha(-\bar{z})}{2}, \\
\ddot{s}_y^{(B_1,b)} &:= \frac{-\sigma_\alpha(-\bar{z}) - z_1}{2}.
\end{aligned}
$$

Type II decomposition of power cone (4)

(iii)

$$
\begin{aligned}
\ddot{x}^{(B_1,c)} &:= \begin{bmatrix} 1 \\ \frac{\mathbf{1}}{\sigma_\alpha(\mathbf{1})} \end{bmatrix} \in \partial \mathcal{K}_\alpha, \\
\ddot{y}^{(B_1,c)} &:= \begin{bmatrix} -1 \\ \frac{\mathbf{1}}{\sigma_\alpha(\mathbf{1})} \end{bmatrix} \in \partial \mathcal{K}_\alpha, \\
\ddot{s}_x^{(B_1,c)} &:= \frac{z_1}{2}, \\
\ddot{s}_y^{(B_1,c)} &:= -\frac{z_1}{2}.
\end{aligned}
$$

Type II decomposition of power cone (5)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(b) *If* $z \in B_2$*, then*

$$z = \begin{cases} \ddot{x}^{(B_2,a)} + (-1) \cdot \ddot{y}^{(B_2,a)}, & \text{if} \quad |I_-| = 0, \\ (-1) \cdot \ddot{x}^{(B_2,b)} + \ddot{y}^{(B_2,b)}, & \text{if} \quad |I_+| = 0, \end{cases}$$

where $\ddot{x}^{(B_2,a)}$, $\ddot{y}^{(B_2,a)}$ *are described as (i), and* $\ddot{x}^{(B_2,b)}$, $\ddot{y}^{(B_2,b)}$ *are described as in (ii).*

(i) Let $\ddot{x}^{(B_2,a)} = (\ddot{x}_1^{(B_2,a)}, \ddot{\bar{x}}^{(B_2,a)})$ and $\ddot{y}^{(B_2,a)} = (\ddot{y}_1^{(B_2,a)}, \ddot{\bar{y}}^{(B_2,a)})$.

$$
\begin{aligned}
\ddot{x}_1^{(B_2,a)} &:= z_1, \\
\ddot{\bar{x}}_j^{(B_2,a)} &:= \begin{cases} \bar{z}_j & \text{if} \quad j \in I_+, \\ 1 & \text{if} \quad j \in I_0 \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\ddot{\bar{x}}_i^{(B_2,a)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if} \quad j = k. \end{cases} \\
\ddot{y}_1^{(B_2,a)} &:= 0, \\
\ddot{\bar{y}}_j^{(B_2,a)} &:= \begin{cases} 0 & \text{if} \quad j \in I_+, \\ 1 & \text{if} \quad j \in I_0 \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\ddot{\bar{x}}_i^{(B_2,a)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if} \quad j = k. \end{cases}
\end{aligned}
$$

(ii) Let $\ddot{x}^{(B_2,b)} = (\ddot{x}_1^{(B_2,b)}, \bar{\ddot{x}}^{(B_2,b)})$ and $\ddot{y}^{(B_2,b)} = (\ddot{y}_1^{(B_2,b)}, \bar{\ddot{y}}^{(B_2,b)})$.

$$
\begin{aligned}
\ddot{x}_1^{(B_2,b)} &:= -z_1, \\
\bar{\ddot{x}}_j^{(B_2,b)} &:= \begin{cases} -\bar{z}_j & \text{if } \; j \in I_-, \\ 1 & \text{if } \; j \in I_0 \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\bar{\ddot{x}}_i^{(B_2,b)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } \; j = k. \end{cases} \\
\ddot{y}_1^{(B_2,b)} &:= 0, \\
\bar{\ddot{y}}_j^{(B_2,b)} &:= \begin{cases} 0 & \text{if } \; j \in I_-, \\ 1 & \text{if } \; j \in I_0 \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\bar{\ddot{x}}_i^{(B_2,b)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } \; j = k. \end{cases}
\end{aligned}
$$

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(c) *If* $z \in B_3$, *then* $z = \ddot{x}^{(B_3)} + (-1) \cdot \ddot{y}^{(B_3)}$, *where* $\ddot{x}^{(B_3)} = (\ddot{x}_1^{(B_3)}, \ddot{\bar{x}}^{(B_3)})$ *and* $\ddot{y}^{(B_3)} = (\ddot{y}_1^{(B_3)}, \ddot{\bar{y}}^{(B_3)})$ *are given by*

$$
\begin{aligned}
\ddot{x}_1^{(B_3)} &:= z_1, \\
\ddot{\bar{x}}_j^{(B_3)} &:= \begin{cases} \bar{z}_j & \text{if } j \in I_+, \\ -\bar{z}_j & \text{if } j \in I_- \text{ and } j \neq k, \\ \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\ddot{\bar{x}}_i^{(B_3)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } j = k. \end{cases} \\
\ddot{y}_1^{(B_3)} &:= 0, \\
\ddot{\bar{y}}_j^{(B_3)} &:= \begin{cases} 0 & \text{if } j \in I_+, \\ -2\bar{z}_j & \text{if } j \in I_- \text{ and } j \neq k, \\ -\bar{z}_k + \left(\dfrac{|z_1|}{\prod_{i \neq k} \left(\ddot{\bar{x}}_i^{(B_3)} \right)^{\alpha_i}} \right)^{\frac{1}{\alpha_k}} & \text{if } j = k. \end{cases}
\end{aligned}
$$

Type II decomposition of power cone (9)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(d) *If $z \in B_4$, then $z = \ddot{x}^{(B_4)} + (-1) \cdot \ddot{y}^{(B_4)}$, where $\ddot{x}^{(B_4)}$ and $\ddot{y}^{(B_4)}$ are given by*

$$\ddot{x}^{(B_4)} = (\ddot{x}_1^{(B_4)}, \ddot{\bar{x}}^{(B_4)}) \quad \text{and} \quad \ddot{y}^{(B_4)} = (\ddot{y}_1^{(B_4)}, \ddot{\bar{y}}^{(B_4)})$$

with

$$\begin{aligned} \ddot{x}_1^{(B_4)} &:= 0, \\ \ddot{\bar{x}}^{(B_4)} &:= \mathbf{1} - \mathbf{1}_k, \\ \ddot{y}_1^{(B_4)} &:= 0, \\ \ddot{\bar{y}}^{(B_4)} &:= \mathbf{1} - \mathbf{1}_k, \end{aligned}$$

and $\mathbf{1}_k$ being the kth column of the identity matrix $I_2 \in \mathbb{R}^{2\times 2}$ $(k = 1, 2)$.

Manipulation of a real example

Example

The power cone $\mathcal{K}_{\frac{1}{2}}$ and its polar cone $\mathcal{K}^{\circ}_{\frac{1}{2}}$ are respectively given by

$$\begin{aligned} \mathcal{K}_{\frac{1}{2}} &= \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, \bar{x}_1^{\frac{1}{2}} \bar{x}_2^{\frac{1}{2}} \geq |x_1|,\ \bar{x}_1 \geq 0,\ \bar{x}_2 \geq 0 \right\}, \\ \mathcal{K}^{\circ}_{\frac{1}{2}} &= \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, (-2\bar{x}_1)^{\frac{1}{2}} (-2\bar{x}_2)^{\frac{1}{2}} \geq |x_1|,\ \bar{x}_1 \leq 0,\ \bar{x}_2 \leq 0 \right\}. \end{aligned}$$

According to the aforementioned four blocks, we pick four different points to figure out their decompositions with respect to $\mathcal{K}_{\frac{1}{2}}$, respectively.

$z = (1,2,2)^T \in \mathbb{R}^3$ in Block I

Let $z = (1,2,2)^T \in \mathbb{R}^3$. In this case, we have $z_1 = 1$ and $\bar{z} = (2,2)^T$, which implies

$$\begin{aligned}
&\bar{z}_{\min} = 2 > 0, \quad \bar{z}_{\max} = 2 > 0, \\
&I_- = \emptyset, \quad I_0 = \emptyset, \quad I_+ = \{1,2\}, \\
&|I_-| = 0, \quad |I_0| = 0, \quad |I_+| = 2, \\
&\sigma_{\frac{1}{2}}(\bar{z}) = 2 > 0, \quad \eta_{\frac{1}{2}}(\bar{z}) = 4 > 0.
\end{aligned}$$

This point $z = (1,2,2)^T$ indeed lies in case (i) of the set B_1.

Type I decompositions of $z = (1,2,2)^T$

$$\dot{x}^{(B_1,a)} = \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_{\frac{1}{2}}(\bar{z})} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\dot{y}^{(B_1,a)} = \begin{bmatrix} 1 \\ -\frac{\bar{z}}{\eta_{\frac{1}{2}}(\bar{z})} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix},$$

$$\dot{s}_x^{(B_1,a)} := \frac{z_1 + \eta_\alpha(\bar{z})}{\sigma_\alpha(\bar{z}) + \eta_\alpha(\bar{z})} \cdot \sigma_\alpha(\bar{z}) = \frac{1+4}{2+4} \cdot 2 = \frac{5}{3},$$

$$\dot{s}_y^{(B_1,a)} := \frac{z_1 - \sigma_\alpha(\bar{z})}{\sigma_\alpha(\bar{z}) + \eta_\alpha(\bar{z})} \cdot \eta_\alpha(\bar{z}) = \frac{1-2}{2+4} \cdot 4 = -\frac{2}{3}.$$

The Type I decompositions of $z = (1,2,2)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-\frac{2}{3}) \cdot \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

Type II decompositions of $z = (1,2,2)^T$

$$\ddot{x}^{(B_1,a)} = \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_{\frac{1}{2}}(\bar{z})} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\ddot{y}^{(B_1,a)} = \begin{bmatrix} -1 \\ \frac{\bar{z}}{\sigma_{\frac{1}{2}}(\bar{z})} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

$$\ddot{s}_x^{(B_1,a)} := \frac{z_1 + \sigma_\alpha(\bar{z})}{2} = \frac{1+2}{2} = \frac{3}{2},$$

$$\ddot{s}_y^{(B_1,a)} := \frac{\sigma_\alpha(\bar{z}) - z_1}{2} = \frac{2-1}{2} = \frac{1}{2}.$$

The Type II decompositions of $z = (1,2,2)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$z = (1, 0, 1)^T \in \mathbb{R}^3$ in Block II

Let $z = (1, 0, 1)^T \in \mathbb{R}^3$. In this case, we have $z_1 = 1$ and $\bar{z} = (0, 1)^T$, which implies

$$\begin{aligned} &\bar{z}_{\min} = 0, \quad \bar{z}_{\max} = 1 > 0, \\ &I_- = \emptyset, \quad I_0 = \{1\}, \quad I_+ = \{2\}, \\ &|I_-| = 0, \quad |I_0| = 1, \quad |I_+| = 1. \end{aligned}$$

This point $z = (1, 0, 1)^T$ indeed lies in case (i) of the set B_2.

Type I decompositions of $z = (1,0,1)^T$

$$
\begin{aligned}
\dot{x}^{(B_2,a)} &= (\dot{x}_1^{(B_2,a)}, \dot{\bar{x}}^{(B_2,a)}),\ \dot{x}_1^{(B_2,a)} = 1,\ \dot{\bar{x}}^{(B_2,a)} = (1,1)^T, \\
s_x &= 1, \\
\dot{y}^{(B_2,a)} &= (\dot{y}_1^{(B_2,a)}, \dot{\bar{y}}^{(B_2,a)}),\ \dot{y}_1^{(B_2,a)} = 0,\ \dot{\bar{y}}^{(B_2,a)} = (-1,0)^T, \\
s_y &= 1.
\end{aligned}
$$

The Type I decompositions of $z = (1,0,1)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$
\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.
$$

Type II decompositions of $z = (1, 0, 1)^T$

$$
\begin{array}{lcl}
\ddot{x}^{(B_2,a)} & = & (\ddot{x}_1^{(B_2,a)}, \ddot{\bar{x}}^{(B_2,a)}),\ \ddot{x}_1^{(B_2,a)} = 1,\ \ddot{\bar{x}}^{(B_2,a)} = (1,1)^T, \\
s_x & = & 1, \\
\ddot{y}^{(B_2,a)} & = & (\ddot{y}_1^{(B_2,a)}, \ddot{\bar{y}}^{(B_2,a)}),\ \ddot{y}_1^{(B_2,a)} = 0,\ \ddot{\bar{y}}^{(B_2,a)} = (1,0)^T, \\
s_y & = & -1.
\end{array}
$$

The Type II decompositions of $z = (1, 0, 1)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$
\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
$$

$z = (1, 1, -1)^T \in \mathbb{R}^3$ in Block III

Let $z = (1, 1, -1)^T \in \mathbb{R}^3$. In this case, we have $z_1 = 1$ and $\bar{z} = (1, -1)^T$, which implies

$$\bar{z}_{\min} = -1 < 0, \quad \bar{z}_{\max} = 1 > 0,$$
$$I_- = \{2\}, \quad I_0 = \emptyset, \quad I_+ = \{1\},$$
$$|I_-| = 1, \quad |I_0| = 0, \quad |I_+| = 1.$$

This point $z = (1, 1, -1)^T$ indeed lies in the set B_3.

$$
\begin{array}{lcl}
\dot{x}^{(B_3)} & = & (\dot{x}_1^{(B_3)}, \dot{\bar{x}}^{(B_3)}),\ \ \dot{x}_1^{(B_3)} = 1,\ \ \dot{\bar{x}}^{(B_3)} = (1,1)^T, \\
s_x & = & 1, \\
\dot{y}^{(B_3)} & = & (\dot{y}_1^{(B_3)}, \dot{\bar{y}}^{(B_3)}),\ \ \dot{y}_1^{(B_3)} = 0,\ \ \dot{\bar{y}}^{(B_3)} = (0,-2)^T, \\
s_y & = & 1.
\end{array}
$$

The Type I decompositions of $z = (1,1,-1)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$
\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.
$$

Type II decompositions of $z = (1, 1, -1)^T$

$$\begin{array}{lcl} \ddot{x}^{(B_3)} & = & (\ddot{x}_1^{(B_3)}, \ddot{\bar{x}}^{(B_3)}),\ \ddot{x}_1^{(B_3)} = 1,\ \ddot{\bar{x}}^{(B_3)} = (1,1)^T, \\ s_x & = & 1, \\ \ddot{y}^{(B_3)} & = & (\ddot{y}_1^{(B_3)}, \ddot{\bar{y}}^{(B_3)}),\ \ddot{y}_1^{(B_3)} = 0,\ \ddot{\bar{y}}^{(B_3)} = (0,2)^T, \\ s_y & = & -1. \end{array}$$

The Type II decompositions of $z = (1, 1, -1)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

$z = (0,0,0)^T \in \mathbb{R}^3$ in Block IV

Let $z = (0,0,0)^T \in \mathbb{R}^3$. In this case, z indeed lies in the set B_4.

Type I decompositions of $z = (0,0,0)^T$

$$\dot{x}^{(B_4)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \dot{y}^{(B_4)} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

or

$$\dot{x}^{(B_4)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \dot{y}^{(B_4)} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

The Type I decompositions of $z = (0,0,0)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Type II decompositions of $z = (0,0,0)^T$

$$\ddot{x}^{(B_4)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \ddot{y}^{(B_4)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or

$$\ddot{x}^{(B_4)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \ddot{y}^{(B_4)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The Type II decompositions of $z = (0,0,0)^T$ with respect to $\mathcal{K}_{\frac{1}{2}}$ is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The dual of exponential cone $\mathcal{K}_{\exp}$

Now, we turn to the exponential cone $\mathcal{K}_{\exp}$, which is defined as

$$\mathcal{K}_{\exp} := \mathrm{cl}\left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, \bar{x}_2 \cdot \exp\left(\frac{\bar{x}_1}{\bar{x}_2}\right) \le x_1,\ \bar{x}_2 > 0,\ x_1 \ge 0\right\}.$$

The dual of the exponential cone $\mathcal{K}_{\exp}$ (denoted by $\mathcal{K}^*_{\exp}$) is described in the form of

$$\mathcal{K}^*_{\exp} := \mathrm{cl}\left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,\middle|\, -\frac{\bar{x}_1}{e} \cdot \exp\left(\frac{\bar{x}_2}{\bar{x}_1}\right) \le x_1,\ \bar{x}_1 < 0,\ x_1 \ge 0\right\}.$$

The boundary of $\mathcal{K}_{\exp}$ and $\mathcal{K}^*_{\exp}$ (1)

Denote $\sigma_{\exp} : \mathbb{R} \times \{\mathbb{R} \setminus \{0\}\} \to \mathbb{R}$ and $\eta_{\exp} : \{\mathbb{R} \setminus \{0\}\} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\sigma_{\exp}(\bar{x}) := \bar{x}_2 \cdot \exp\left(\frac{\bar{x}_1}{\bar{x}_2}\right), \quad \eta_{\exp}(\bar{x}) := \frac{\bar{x}_1}{e} \cdot \exp\left(\frac{\bar{x}_2}{\bar{x}_1}\right).$$

Then, the boundary of $\mathcal{K}_{\exp}$ and $\mathcal{K}^*_{\exp}$ (denoted by $\partial\mathcal{K}_{\exp}$ and $\partial\mathcal{K}^*_{\exp}$) are respectively given by

$$\begin{aligned} \partial\mathcal{K}_{\exp} &:= S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{0\}, \\ \partial\mathcal{K}^*_{\exp} &:= S_5 \cup S_6 \cup S_7 \cup S_8 \cup \{0\}, \end{aligned}$$

where the sets S_i $(i = 1, 2, \cdots, 8)$ are defined by

$$
\begin{aligned}
S_1 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 > 0,\ \bar{x}_1 < 0,\ \bar{x}_2 = 0\right\}, \\
S_2 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 < 0,\ \bar{x}_2 = 0\right\}, \\
S_3 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 > 0,\ \bar{x}_1 = 0,\ \bar{x}_2 = 0\right\}, \\
S_4 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 \geq 0,\ \sigma_{\exp}(\bar{x}) = x_1,\ \bar{x}_2 > 0\right\}, \\
S_5 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 > 0,\ \bar{x}_1 = 0,\ \bar{x}_2 > 0\right\}, \\
S_6 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 = 0,\ \bar{x}_2 > 0\right\}, \\
S_7 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 > 0,\ \bar{x}_1 = 0,\ \bar{x}_2 = 0\right\}, \\
S_8 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 \geq 0,\ \bar{x}_1 < 0,\ -\eta_{\exp}(\bar{x}) = x_1\right\}.
\end{aligned}
$$

The polar of exponential cone $\mathcal{K}_{\exp}$

The polar of $\mathcal{K}_{\exp}$ (denoted by $\mathcal{K}_{\exp}^{\circ}$) is characterized as

$$\mathcal{K}_{\exp}^{\circ} := \text{cl}\left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \;\middle|\; -\frac{\bar{x}_1}{e} \cdot \exp\left(\frac{\bar{x}_2}{\bar{x}_1}\right) \geq x_1,\ \bar{x}_1 > 0,\ x_1 \leq 0\right\}$$

and its boundary is given by

$$\partial\mathcal{K}_{\exp}^{\circ} := T_1 \cup T_2 \cup T_3 \cup T_4 \cup \{0\},$$

where the set T_i $(i = 1, 2, 3, 4)$ are described as follows:

$$\begin{aligned}
T_1 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 < 0,\ \bar{x}_1 = 0,\ \bar{x}_2 < 0\right\},\\
T_2 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 = 0,\ \bar{x}_1 = 0,\ \bar{x}_2 < 0\right\},\\
T_3 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 < 0,\ \bar{x}_1 = 0,\ \bar{x}_2 = 0\right\},\\
T_4 &:= \left\{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 \leq 0,\ \bar{x}_1 > 0,\ -\eta_{\exp}(\bar{x}) = x_1\right\}.
\end{aligned}$$

The set $\mathcal{K}_{\exp} \cup \mathcal{K}_{\exp}^{\circ}$ can be divided into the following nine parts

$$\mathcal{K}_{\exp} \cup \mathcal{K}_{\exp}^{\circ} = S_1 \cup S_2 \cup S_3 \cup T_1 \cup T_2 \cup T_3 \cup P_1 \cup P_2 \cup \{0\},$$

with the sets P_1 and P_2 given by

$$P_1 := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 \geq 0,\ \sigma_{\exp}(\bar{x}) \leq x_1,\ \bar{x}_2 > 0 \right\},$$
$$P_2 := \left\{ (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, x_1 \leq 0,\ \bar{x}_1 > 0,\ -\eta_{\exp}(\bar{x}) \geq x_1 \right\}.$$

Four blocks for exponential cone setting: Type I (1)

Again, the key to deriving Type I decomposition of exponential cone $\mathcal{K}_{\exp}$ is dividing the space $\mathbb{R} \times \mathbb{R}^2$ into the following four blocks:

Block I:

$$\tilde{B}_1 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_1 \cdot \bar{z}_2 > 0 \text{ or } (\bar{z} = 0 \text{ and } z_1 \neq 0) \right\}.$$

The set $\tilde{B}_1$ includes three subcases: (i) $\bar{z}_1 > 0, \ \bar{z}_2 > 0$; (ii) $\bar{z}_1 < 0, \ \bar{z}_2 < 0$; (iii) $\bar{z} = 0, \ z_1 \neq 0$, where $\bar{z} := (\bar{z}_1, \bar{z}_2)^T \in \mathbb{R}^2$.

Block II:

$$\tilde{B}_2 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, (\bar{z}_1 = 0, \ \bar{z}_2 \neq 0) \text{ or } (\bar{z}_1 < 0, \ \bar{z}_2 > 0) \right\}.$$

The set $\tilde{B}_2$ consists of the points in the following three subcases: (i) $\bar{z}_1 = 0, \ \bar{z}_2 > 0$; (ii) $\bar{z}_1 = 0, \ \bar{z}_2 < 0$; (iii) $\bar{z}_1 < 0, \ \bar{z}_2 > 0$.

Block III:

$$\tilde{B}_3 := \left\{(z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, (\bar{z}_1 \neq 0,\ \bar{z}_2 = 0) \ \text{ or } (\bar{z}_1 > 0,\ \bar{z}_2 < 0)\right\}.$$

Like the set $\tilde{B}_2$, this set also includes three subcases: (i) $\bar{z}_1 > 0,\ \bar{z}_2 = 0$; (ii) $\bar{z}_1 < 0,\ \bar{z}_2 = 0$; (iii) $\bar{z}_1 > 0,\ \bar{z}_2 < 0$.

Block IV:

$$\tilde{B}_4 := \left\{(z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z} = 0 \text{ and } z_1 = 0\right\}.$$

This set includes only one point $(0, 0) \in \mathbb{R} \times \mathbb{R}^2$.

Type I decomposition of exponential cone $\mathcal{K}_{\exp}$ (1)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

For any given $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2$, the Type I decomposition of the exponential cone $\mathcal{K}_{\exp}$ is described as follows:

(a) *If $z \in \tilde{B}_1$, then*

$$z = \begin{cases} \hat{s}_x^{(\tilde{B}_1,a)} \cdot \hat{x}^{(\tilde{B}_1,a)} + \hat{s}_y^{(\tilde{B}_1,a)} \cdot \hat{y}^{(\tilde{B}_1,a)}, & \text{if } \bar{z} \neq 0, \\ \hat{s}_x^{(\tilde{B}_1,b)} \cdot \hat{x}^{(\tilde{B}_1,b)} + \hat{s}_y^{(\tilde{B}_1,b)} \cdot \hat{y}^{(\tilde{B}_1,b)}, & \text{if } \bar{z} = 0, \end{cases}$$

where $\hat{x}^{(\tilde{B}_1,a)}$, $\hat{y}^{(\tilde{B}_1,a)}$, $\hat{s}_x^{(\tilde{B}_1,a)}$, $\hat{s}_y^{(\tilde{B}_1,a)}$ are defined as in (i), and $\hat{x}^{(\tilde{B}_1,b)}$, $\hat{y}^{(\tilde{B}_1,b)}$, $\hat{s}_x^{(\tilde{B}_1,b)}$, $\hat{s}_y^{(\tilde{B}_1,b)}$ are defined as in (ii).

(i)

$$\begin{aligned}
\hat{x}^{(\tilde{B}_1,a)} &:= \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_{\exp}(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_{\exp}, \\
\hat{y}^{(\tilde{B}_1,a)} &:= \begin{bmatrix} -1 \\ \frac{\bar{z}}{\eta_{\exp}(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}^{\circ}_{\exp}, \\
\hat{s}_x^{(\tilde{B}_1,a)} &:= \frac{z_1 + \eta_{\exp}(\bar{z})}{\sigma_{\exp}(\bar{z}) + \eta_{\exp}(\bar{z})} \cdot \sigma_{\exp}(\bar{z}), \\
\hat{s}_y^{(\tilde{B}_1,a)} &:= \frac{\sigma_{\exp}(\bar{z}) - z_1}{\sigma_{\exp}(\bar{z}) + \eta_{\exp}(\bar{z})} \cdot \eta_{\exp}(\bar{z}).
\end{aligned}$$

Type I decomposition of exponential cone $\mathcal{K}_{\exp}$ (3)

(ii) Denote $\mathbf{1} := (1,1)^T \in \mathbb{R}^2$.

$$\hat{x}^{(\tilde{B}_1,b)} := \begin{bmatrix} 1 \\ \frac{\mathbf{1}}{\sigma_{\exp}(\mathbf{1})} \end{bmatrix} \in \partial\mathcal{K}_{\exp},$$

$$\hat{y}^{(\tilde{B}_1,b)} := \begin{bmatrix} -1 \\ \frac{\mathbf{1}}{\eta_{\exp}(\mathbf{1})} \end{bmatrix} \in \partial\mathcal{K}^{\circ}_{\exp},$$

$$\hat{s}_x^{(\tilde{B}_1,b)} := \frac{z_1}{\sigma_{\exp}(\mathbf{1}) + \eta_{\exp}(\mathbf{1})} \cdot \sigma_{\exp}(\mathbf{1}),$$

$$\hat{s}_y^{(\tilde{B}_1,b)} := \frac{-z_1}{\sigma_{\exp}(\mathbf{1}) + \eta_{\exp}(\mathbf{1})} \cdot \eta_{\exp}(\mathbf{1}).$$

Type I decomposition of exponential cone $\mathcal{K}_{\exp}$ (4)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(b) *If* $z \in \tilde{B}_2$, *then* $z = \hat{s}_x^{(\tilde{B}_2)} \cdot \hat{x}^{(\tilde{B}_2)} + \hat{s}_y^{(\tilde{B}_2)} \cdot \hat{y}^{(\tilde{B}_2)}$, *where* $\hat{x}^{(\tilde{B}_2)}$, $\hat{y}^{(\tilde{B}_2)}$, $\hat{s}_x^{(\tilde{B}_2)}$, *and* $\hat{s}_y^{(\tilde{B}_2)}$ *are given by*

$$\begin{aligned} \hat{x}^{(\tilde{B}_2)} &:= \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_{\exp}(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_{\exp}, \\ \hat{y}^{(\tilde{B}_2)} &:= \begin{bmatrix} -|z_1 - \sigma_{\exp}(\bar{z})| \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp}^{\circ}, \\ \hat{s}_x^{(\tilde{B}_2)} &:= \sigma_{\exp}(\bar{z}), \\ \hat{s}_y^{(\tilde{B}_2)} &:= sgn(\sigma_{\exp}(\bar{x}) - z_1). \end{aligned}$$

Type I decomposition of exponential cone $\mathcal{K}_{\exp}$ (5)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(c) *If* $z \in \tilde{B}_3$*, then* $z = \hat{s}_x^{(\tilde{B}_3)} \cdot \hat{x}^{(\tilde{B}_3)} + \hat{s}_y^{(\tilde{B}_3)} \cdot \hat{y}^{(\tilde{B}_3)}$*, where* $\hat{x}^{(\tilde{B}_3)}$*,* $\hat{y}^{(\tilde{B}_3)}$*,* $\hat{s}_x^{(\tilde{B}_3)}$*, and* $\hat{s}_y^{(\tilde{B}_3)}$ *are given by*

$$\begin{aligned}
\hat{x}^{(\tilde{B}_3)} &:= \begin{bmatrix} |z_1 + \eta_{\exp}(\bar{z})| \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp}, \\
\hat{y}^{(\tilde{B}_3)} &:= \begin{bmatrix} -1 \\ \frac{\bar{z}}{\eta_{\exp}(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_{\exp}^{\circ}, \\
\hat{s}_x^{(\tilde{B}_3)} &:= sgn(z_1 + \eta_{\exp}(\bar{z})), \\
\hat{s}_y^{(\tilde{B}_3)} &:= \eta_{\exp}(\bar{z}).
\end{aligned}$$

Type I decomposition of exponential cone $\mathcal{K}_{\exp}$ (6)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(d) *If* $z \in \tilde{B}_4$*, then* $z = \hat{x}^{(\tilde{B}_4)} + \hat{y}^{(\tilde{B}_4)}$*, where* $\hat{x}^{(\tilde{B}_4)}$ *and* $\hat{y}^{(\tilde{B}_4)}$ *are given by*

$$\hat{x}^{(\tilde{B}_4)} := \begin{bmatrix} \max\{0, w\} \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp},$$

$$\hat{y}^{(\tilde{B}_4)} := \begin{bmatrix} \min\{0, -w\} \\ 0 \end{bmatrix} \in \partial\mathcal{K}^{\circ}_{\exp},$$

with w *being any scalar in* $\mathbb{R}$.

For deriving Type II decomposition of exponential cone $\mathcal{K}_{\exp}$, another different four blocks for the space $\mathbb{R} \times \mathbb{R}^2$ is needed.

Block I:

$$\bar{B}_1 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_2 \neq 0 \right\}, \text{ where } \bar{z} := (\bar{z}_1, \bar{z}_2)^T \in \mathbb{R}^2.$$

The set $\bar{B}_1$ includes six subcases: (1) $\bar{z}_1 > 0, \ \bar{z}_2 > 0$. (2) $\bar{z}_1 = 0, \ \bar{z}_2 > 0$. (3) $\bar{z}_1 < 0, \ \bar{z}_2 > 0$. (4) $\bar{z}_1 > 0, \ \bar{z}_2 < 0$. (5) $\bar{z}_1 = 0, \ \bar{z}_2 < 0$. (6) $\bar{z}_1 < 0, \ \bar{z}_2 < 0$.

Block II:

$$\bar{B}_2 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, (\bar{z} = 0, \ z_1 \neq 0) \text{ or } (\bar{z}_1 < 0, \ \bar{z}_2 = 0) \right\}.$$

The set $\bar{B}_2$ consists of the points in the following two subcases: (1) $\bar{z}_1 = 0, \ \bar{z}_2 = 0, \ z_1 \neq 0$. (2) $\bar{z}_1 < 0, \ \bar{z}_2 = 0$.

Block III:

$$\bar{B}_3 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z}_1 > 0,\ \bar{z}_2 = 0 \right\}.$$

Similar to the set $\bar{B}_2$, this set also includes the points $(z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2$ that $\bar{z}_1 > 0,\ \bar{z}_2 = 0$.

Block IV:

$$\bar{B}_4 := \left\{ (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{z} = 0 \text{ and } z_1 = 0 \right\}.$$

This set includes only one point $(0, 0) \in \mathbb{R} \times \mathbb{R}^2$.

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

For any given $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2$, the Type II decomposition of the exponential cone $\mathcal{K}_{\exp}$ is described as follows:

(a) *If $z \in \bar{B}_1$, then $z = \check{s}_x^{(\bar{B}_1)} \cdot \check{x}^{(\bar{B}_1)} + \check{s}_y^{(\bar{B}_1)} \cdot \check{y}^{(\bar{B}_1)}$, where $\check{x}^{(\bar{B}_1)}$, $\check{y}^{(\bar{B}_1)}$, $\check{s}_x^{(\bar{B}_1)}$, and $\check{s}_y^{(\bar{B}_1)}$ are given by*

$$\begin{aligned}
\check{x}^{(\bar{B}_1)} &:= \begin{bmatrix} 1 \\ \frac{\bar{z}}{\sigma_{\exp}(\bar{z})} \end{bmatrix} \in \partial\mathcal{K}_{\exp}, \\
\check{y}^{(\bar{B}_1)} &:= \begin{bmatrix} |z_1 - \sigma_{\exp}(\bar{z})| \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp}, \\
\check{s}_x^{(\bar{B}_1)} &:= \sigma_{\exp}(\bar{z}), \\
\check{s}_y^{(\bar{B}_1)} &:= sgn(z_1 - \sigma_{\exp}(\bar{z})).
\end{aligned}$$

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(b) *If* $z \in \bar{B}_2$, *then* $z = \check{x}^{(\bar{B}_2)} + (-1) \cdot \check{y}^{(\bar{B}_2)}$, *where* $\check{x}^{(\bar{B}_2)}$ *and* $\check{y}^{(\bar{B}_2)}$ *are given by*

$$\check{x}^{(\bar{B}_2)} := \begin{bmatrix} \max\{0, z_1\} \\ \bar{z} \end{bmatrix} \in \partial \mathcal{K}_{\exp},$$

$$\check{y}^{(\bar{B}_2)} := \begin{bmatrix} -\min\{0, z_1\} \\ 0 \end{bmatrix} \in \partial \mathcal{K}_{\exp}.$$

Type II decomposition of exponential cone $\mathcal{K}_{\exp}$ (3)

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(c) *If* $z \in \bar{B}_3$, *then* $z = \check{x}^{(\bar{B}_3)} + (-1) \cdot \check{y}^{(\bar{B}_3)}$, *where* $\check{x}^{(\bar{B}_3)}$ *and* $\check{y}^{(\bar{B}_3)}$ *are given by*

$$\begin{aligned} \check{x}^{(\bar{B}_3)} &:= \begin{bmatrix} \max\{0, z_1\} \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp}, \\ \check{y}^{(\bar{B}_3)} &:= \begin{bmatrix} -\min\{0, z_1\} \\ -\bar{z} \end{bmatrix} \in \partial\mathcal{K}_{\exp}. \end{aligned}$$

Theorem (Lu-Yang-Chen-Qi, JOGO, 2020)

(d) *If* $z \in \bar{B}_4$, *then* $z = \check{x}^{(\bar{B}_4)} + (-1) \cdot \check{y}^{(\bar{B}_4)}$, *where* $\check{x}^{(\bar{B}_4)}$ *and* $\check{y}^{(\bar{B}_4)}$ *are given by*

$$\check{x}^{(\bar{B}_4)} := \begin{bmatrix} \max\{0, w\} \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp},$$

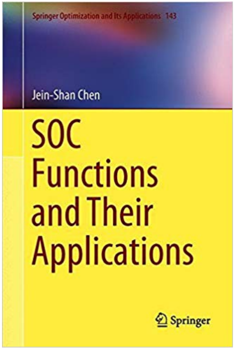
$$\check{y}^{(\bar{B}_4)} := \begin{bmatrix} -\min\{0, -w\} \\ 0 \end{bmatrix} \in \partial\mathcal{K}_{\exp},$$

with w *being any scalar in* $\mathbb{R}$.

Contributions and future directions

- The uniqueness of our decompositions at any nonzero point is a fascinating feature to avoid the hurdle for analyzing theoretical properties of the related conic functions and very helpful to designing numerical algorithms.
- Which type of decomposition is more useful? We guess that Type I may be more helpful for subsequent analysis towards general non-symmetric cones, because the non-symmetric feature is an uncertain factor and increases analysis complexity.
- Future direction: designing the solutions methods by exploiting these decompositions.

The End



∼ Thanks for your attention ∼