

Embedding of partially ordered topological spaces in Fell topological hyperspaces

Jinlu Li

Department of mathematics, Shawnee State University, Portsmouth Ohio USA

Joint work with Ching-Feng Wen, Jen-Chin Yao and Congjun Zhang

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Partially ordered topological spaces

Let (X, τ) be a topological space equipped with a partial order \preceq . For every $x \in X$, we denote the \preceq -principal ideal and principal filter of x in (X, \preceq) , respectively, as follows

$$x^\downarrow = \{u \in X: u \preceq x\} \quad \text{and} \quad x^\uparrow = \{u \in X: u \succeq x\}.$$

Let A be a nonempty subset of X , we write

$$A^\downarrow = \{u \in X: u \preceq x, \text{ for some } x \in A\},$$

and

$$A^\uparrow = \{u \in X: u \succeq x, \text{ for some } x \in A\}.$$

For any $A \subseteq X$, we have $A \subseteq A^\downarrow \cap A^\uparrow$.

If $A^\downarrow = A$, then A is said to be \preceq -decreasing;

If $A^\uparrow = A$, then A is said to be \preceq -increasing.

Partially ordered topological spaces

Let (X, τ) be a topological space equipped with a partial order \preceq . If \preceq is a closed subset in $X \times X$, then (X, τ, \preceq) is called a **partially ordered topological space**.

Proposition 2.1 (*topology and order by Nachbin*) *Let (X, τ) be a topological space equipped with a partial order \preceq . Then the following two properties are equivalent:*

- (i) (X, τ, \preceq) is a partially ordered topological space ;
- (ii) For $x, y \in X$, whenever $x \not\preceq y$ false, there exist disjoint neighborhoods V of x and U of y such that V is \preceq -increasing and U is \preceq -decreasing.

Moreover, (ii) implies the following (iii) and (iii) does not imply (ii).

- (iii) For every $x \in X$, both x^\downarrow and x^\uparrow are τ -closed.

Topologies on the hyperspaces of partially ordered topological spaces

Let (X, τ, \preceq) be a partially ordered topological space. Let $C(X)$ denote the collection of all τ -closed subsets of X and $C_0(X) = C(X) \setminus \{\emptyset\}$. Denote the set of all \preceq -principal ideals in X by

$$C^\downarrow(X) = \{x^\downarrow : x \in X\}.$$

By Part (iii) in Proposition 2.1 (Nachbin), we have $C^\downarrow(X) \subseteq C(X)$.

We recall several topologies defined on $C(X)$

The Fell topology τ_F on $C(X)$

For every τ -open subset O of X and every τ -compact subsets D of X , we denote the following subsets of $C(X)$:

$$O^- = \{A \in C(X) : A \cap O \neq \emptyset\} \quad \text{and} \quad (X \setminus D)^+ = \{A \in C(X) : A \cap D = \emptyset\}.$$

The set of subsets O^- for O running through the collection of all τ -open subsets and $(X \setminus D)^+$ for D running through all τ -compact subsets of X form the base of a topology on $C(X)$, which is called the Fell topology on $C(X)$ and it is denoted by τ_F .

Therefore, $(C(X), \tau_F)$ is called the Fell topological hyperspace induced by this topological space (X, τ) .

Then $(C_0(X), \tau_F)$ and $(C^\downarrow(X), \tau_F)$ are also topological spaces with relative topology τ_F .

Some properties of the Fell topological hyperspace $(C(X), \tau_F)$

Let (X, τ) be a Hausdorff topological space. The Fell topological hyperspace $(C(X), \tau_F)$ has many useful properties (see the book by Beer and a paper by Beer and Ok). We list two of them below

- (F₁) $(C(X), \tau_F)$ is compact;
- (F₂) $x \rightarrow \{x\}$ (topologically) embeds X in $C(X)$.

G. Beer, Metric spaces with nice closed balls and distance functions for closed sets,
Bull. Austral. Math. Soc. 35 (1987), 81-96.

G. Beer and E. A. Ok, Embedding of topological posets in hyperspaces, arXiv:2111.11575v1 [math.GN], Nov. 2021.

The Vietoris topology τ_V on $C(X)$.

The Vietoris topology τ_V on $C(X)$ is induced by the following base

$$\{O^- : O \text{ is an } \tau\text{-open subset of } X\} \cup \{(X \setminus E)^+ : E \text{ is a } \tau\text{-closed subset of } X\}.$$

Then $(C(X), \tau_V)$ is called the Vietoris topological hyperspace of (X, τ) .

The Hausdorff topology τ_H on $C(X)$.

In particular, let (X, τ) be a metric space and the topology τ is induced by a metric d on X . The Hausdorff metric H on $C(X)$ is defined, for any distinct $A, B \in C(X)$ as

$$H(A, B) = \max \left\{ \sup_{a \in A} \left(\inf_{b \in B} d(a, b) \right), \sup_{b \in B} \left(\inf_{a \in A} d(b, a) \right) \right\}.$$

Then the Hausdorff metric H induces the Hausdorff topology on $C(X)$, denoted by τ_H and $(C(X), \tau_H)$ is called the Hausdorff topological hyperspace of (X, τ) .

The natural partial order \subseteq on $(C(X), \tau_F)$ (inclusion ordering)

Proposition (Beer and Ok). *Let (X, τ, \leq) be a Hausdorff topological space. Then the following conditions are equivalent:*

- (i) X is locally compact;*
- (ii) \subseteq is closed in $C(X) \times C(X)$ with the Fell topology;*
- (iii) \subseteq is closed in $C_0(X) \times C_0(X)$ with the relative Fell topology.*

The canonical map $x \rightarrow x^\downarrow$

Let (X, τ, \preceq) be a **partially ordered topological space**. In this paper, we consider some properties of the canonical map $x \rightarrow x^\downarrow$ which is from X to $C(X)$ with respect to the above three topologies τ_F , τ_V , and τ_H on $C(X)$.

When (X, τ, \preceq) is a partially ordered Hausdorff topological space, we have

$$\{x\} \in C(X), \text{ for every } x \in X.$$

The property (F_2) states that the map $x \rightarrow \{x\}$ embeds (X, τ) in $(C(X), \tau_F)$.

Order-embedding of (X, τ, \preceq) in $(C(X), \tau_F, \subseteq)$

In a partially ordered topological space (X, τ, \preceq) , $C^\downarrow(X) \subseteq C(X) \setminus \{\emptyset\}$ and $(C^\downarrow(X), \tau_F)$ is a topological subspace of $(C(X), \tau_F)$ with respect to the relative Fell topology.

Let $(C(X), \tau_F)$ be equipped with the natural partial order \subseteq (inclusion ordering). Since \preceq and \subseteq are partial orders on X and $C(X)$, respectively, the canonical map is an order-embedding map, which satisfies that, for $x, y \in X$,

$$E_1. \quad x \preceq y \text{ if and only if } x^\downarrow \subseteq y^\downarrow;$$

$$E_2. \quad x = y \text{ if and only if } x^\downarrow = y^\downarrow.$$

Topologically order-embedding of (X, τ, \preceq) in $(C(X), \tau_F, \subseteq)$

We say that the canonical map $x \rightarrow x^\downarrow$ topologically order-embeds (X, τ, \preceq) in $(C(X), \tau_F, \subseteq)$, whenever, in addition to E_1 and E_2 :

E_1 . $x \preceq y$ if and only if $x^\downarrow \subseteq y^\downarrow$;

E_2 . $x = y$ if and only if $x^\downarrow = y^\downarrow$,

this map $x \rightarrow x^\downarrow$ is also a homeomorphism between (X, τ) and $(C^\downarrow(X), \tau_F)$.

More precisely speaking, the following two conditions are also satisfied

E_3 . The canonical map $x \rightarrow x^\downarrow$ is continuous from (X, τ) to $(C^\downarrow(X), \tau_F)$;

E_4 . The map $x^\downarrow \rightarrow x$ is continuous from $(C^\downarrow(X), \tau_F)$ to (X, τ) .

The topologically order-embedding of partially ordered topological spaces in hyperspaces and its applications have been studied by many authors.

In particular, many authors studied the topologically order-embedding of topological lattices in hyperspaces.

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Beer and Ok recently proved an embedding theorem on Hausdorff topological Λ -semilattices.

G. Beer and E. A. Ok, Embedding of topological posets in hyperspaces, arXiv:2111.11575v1 [math.GN], Nov. 2021.

Theorem (Beer and Ok).

Let (X, τ, \leq) be a locally compact and order-connected Hausdorff topological Λ -semilattice. Then the canonical map $x \rightarrow x^\downarrow$ topologically order-embeds (X, τ, \leq) in $(C(X), \tau_F, \subseteq)$.

Embedding of partially ordered topological spaces in Fell topological hyperspaces

We introduce some concepts with respect to partially ordered topological spaces, which are used to prove the topologically embedding property of the order-embedding canonical map.

Definition. Let (X, τ, \preceq) be a partially ordered topological space. If, for any $x \in X$ and for any open subset O in X ,

$$x^\downarrow \cap O \neq \emptyset$$

implies that there is an open neighborhood D of x such that

$$y^\downarrow \cap O \neq \emptyset, \text{ for any } y \in D,$$

then \preceq is said to be continuously decreasing on (X, τ) .

Proposition. *Let (X, τ, \preceq) be a partially ordered Hausdorff topological space. If \preceq is continuously decreasing on (X, τ) , then the canonical map $x \rightarrow x^\downarrow$ from (X, τ) to $(C^\downarrow(X), \tau_F)$ is continuous.*

Definition (Proper inclusion property)

Let (X, τ, \preceq) be a partially ordered topological space. If, for any $x, y \in X$ with $y \preceq x$,

$$y \in \text{int } x^\downarrow \implies y^\downarrow \subseteq \text{int } x^\downarrow \quad \text{and} \quad x \in \text{int } y^\uparrow \implies x^\uparrow \subseteq \text{int } y^\uparrow,$$

then \preceq is said to have proper inclusion property.

Definition (Dense boundaries)

If, for any $x \in X$ and any open neighborhood O of x , there are $a, b \in X$ with $b \preceq a$ and $x \in \text{int}(b^\uparrow \cap a^\downarrow)$ such that, for any $y \in X$ with $y \notin b^\uparrow \cap a^\downarrow$ and for any $z \in \text{int}(b^\uparrow \cap a^\downarrow)$, we have that

- (a) $y \preceq z$ implies that there is $u \in \partial(b^\uparrow \cap a^\downarrow)$ such that $y \preceq u \preceq z$;
- (b) $z \preceq y$ implies that there is $v \in \partial(b^\uparrow \cap a^\downarrow)$ such that $z \preceq v \preceq y$,

then, (X, τ, \preceq) is said to have dense boundaries.

Definition (Singular point)

Let $x \in X$. If, for any open neighborhood O of x , there is an open neighborhood O_1 of x with $O_1 \subseteq O$, such that

$$\{y \in X \setminus O_1 : u \preceq y, \text{ for some } u \in O_1\} = \emptyset,$$

then, x is called an \preceq -upper singular point.

Definition (Upper compact bounded point)

Let $x \in X$. If, for any open neighborhood O of x , there are points $a, b \in O$ with $b \preceq a$ such that

$$b^\uparrow \cap a^\downarrow \subseteq O, \quad x \in \text{int}(b^\uparrow \cap a^\downarrow)$$

and $(b^\uparrow \cap a^\downarrow) \setminus (\text{int } a^\downarrow)$ is nonempty and compact,

then, x is called an \preceq -upper compact bounded point.

Proposition. *Let (X, τ, \leq) be a partially ordered Hausdorff topological space. Suppose that (X, τ, \leq) satisfies the following conditions:*

- (i) It has proper inclusion property;*
- (ii) It has dense boundaries;*
- (iii) For any point $x \in X$, x is either an \leq -upper singular point, or an \leq -upper compact bounded point.*

Then the map $x^\downarrow \rightarrow x$ is continuous from $(C^\downarrow(X), \tau_F)$ to (X, τ) .

We provide a counter example below to show that condition (iii) in above proposition is a necessary condition to assure the continuity of the map $x^\downarrow \rightarrow x$, from $(C^\downarrow(X), \tau_F)$ to (X, τ) .

Example. Define

$$X = \{(u, v) \in \mathbb{R}^2: uv > 0\} \cup \{\theta\}, \quad \text{where } \theta = (0, 0).$$

Let X be equipped with the standard Euclidean topology τ and equipped with the coordinate wise partial order \preceq . Then (X, τ, \preceq) is a partially ordered topological space. Moreover,

- (a) θ is neither an \preceq -upper singular point, nor an \preceq -upper compact bounded point;
- (b) The map $x^\downarrow \rightarrow x$ from $(C^\downarrow(X), \tau_F)$ to (X, τ) is not continuous at θ^\downarrow .

Theorem (the main theorem in this paper). *Let (X, τ, \preceq) be a partially ordered Hausdorff topological space with \preceq being continuously decreasing. Suppose that (X, τ, \preceq) satisfies the following conditions:*

- (i) It has proper inclusion property;*
- (ii) It has dense boundaries;*
- (iii) For any point $x \in X$, x is either an \preceq -upper singular point, or an \preceq -upper compact bounded point.*

Then map $x \rightarrow x^\downarrow$ topologically order-embeds (X, τ, \preceq) in $(C(X), \tau_F, \subseteq)$.

Corollary. *Let (X, τ, \leq) be a finite-dimensional partially ordered topological vector space. Then the canonical map $x \rightarrow x^\downarrow$ topologically order-embeds (X, τ, \leq) in $(C(X), \tau_F, \subseteq)$.*

Applications to subsets of finite-dimensional partially ordered topological vector spaces

Based on our motivation of applying the embedding theory to vector variational analysis, we study the properties of hyperspaces of finite-dimensional partially ordered topological vector spaces.

Let (X, τ, \preceq) be a given finite-dimensional partially ordered topological vector space. By the above Corollary, it satisfies that the canonical order-embedding is a topological embedding from (X, τ, \preceq) to $(C(X), \tau_F, \subseteq)$.

If we take an arbitrary subset $Y \subseteq X$ satisfying that the subspace (Y, τ, \preceq) is a partially ordered topological space, in which τ is the relative topology restricted on Y , and the hyperspace space $(C(Y), \tau_F, \subseteq)$ is equipped with the relative Fell topology restricted on $C(Y)$.

Then, the canonical map $y \rightarrow y^\downarrow$ is always an order-embedding map from (Y, τ, \preceq) to $(C(Y), \tau_F, \subseteq)$. Then one may ask the following

Question:

Does the canonical map $y \rightarrow y^\downarrow$ topologically embed (Y, τ) in $(C(Y), \tau_F)$?

It is a little surprising to see that, in general, the answer is negative. Moreover, with the relative topology τ_F on $C^\downarrow(Y)$, we have

- (a) The canonical map $y \rightarrow y^\downarrow$ from (Y, τ) to $(C^\downarrow(Y), \tau_F)$ may not be continuous;
- (b) The map $y^\downarrow \rightarrow y$ from $(C^\downarrow(Y), \tau_F)$ to (Y, τ) may not be continuous.

Let \mathbb{R}^n denote the n -d Euclidean space, for some $n > 1$, equipped with the standard topology τ .

Let K be a pointed closed and convex cone in \mathbb{R}^n with nonempty interior, which induces a partial order \preceq on \mathbb{R}^n .

Then $(\mathbb{R}^n, \tau, \preceq)$ is a finite-dimensional partially ordered topological vector space, and the canonical map $x \rightarrow x^\downarrow$ topologically order-embeds $(\mathbb{R}^n, \tau, \preceq)$ in $(C(\mathbb{R}^n), \tau_F, \subseteq)$.

Lemma 1. *For any $a \in K$, $a^\downarrow \cap (-a)^\uparrow$ is bounded.*

Lemma 2. *For any $x \in \mathbb{R}^n$ and any open neighborhood U of x , there are points $a, b \in U$ with $b \preceq x \preceq a$, such that $x \in \text{int}(a^\downarrow \cap b^\uparrow) \subseteq a^\downarrow \cap b^\uparrow \subseteq U$.*

Lemma 3. *For any $a, b, c \in \mathbb{R}^n$ with $b \preceq a$, suppose that $c \notin a^\downarrow \cap b^\uparrow$. Then we have*

- (a) If $c \preceq x$, for some $x \in \text{int}(a^\downarrow \cap b^\uparrow)$, then there is $y \in \partial(a^\downarrow \cap b^\uparrow)$ such that $c \preceq y \preceq x$;*
- (b) If $x \preceq c$, for some $x \in \text{int}(a^\downarrow \cap b^\uparrow)$, then there is $z \in \partial(a^\downarrow \cap b^\uparrow)$ such that $x \preceq z \preceq c$.*

Theorem. *Let Y be a nonempty open subset of \mathbb{R}^n equipped with a partial order \preceq induced by a pointed closed and convex cone K in \mathbb{R}^n with nonempty interior. Then*

- (i) (Y, τ, \preceq) is a partially ordered topological space;*
- (ii) the map $y \rightarrow y^\downarrow$ topologically order-embeds (Y, τ, \preceq) in $(C(Y), \tau_F, \subseteq)$.*

In contrast to the embedding property on the open subsets of \mathbb{R}^n , the situations of closed subsets are very different. We provide next example of closed and locally compact subset Y of \mathbb{R}^3 such that the canonical map $y \rightarrow y^\downarrow$ from (Y, τ, \preceq) to $(C^\downarrow(Y), \tau_F, \subseteq)$ is not continuous.

Special partially ordered topological spaces: topological \wedge -semilattices, topological \vee -semilattices and topological lattices

Let (X, τ, \preceq) be a partially ordered topological space. In addition,

- (a) if \preceq is an \wedge -semilattice and $\wedge: X \times X \rightarrow X$ is a continuous operator, then (X, τ, \preceq) is called a topological \wedge -semilattice;
- (b) if \preceq is a \vee -semilattice and $\vee: X \times X \rightarrow X$ is a continuous operator, then (X, τ, \preceq) is called a topological \vee -semilattice;
- (c) if (X, τ, \preceq) is both topological \wedge -semilattice and topological \vee -semilattice, then (X, τ, \preceq) is called a topological lattice.

Example. Define two closed triangles T_1 and T_2 in \mathbb{R}^3 by

T_1 has vertexes $(0, 0, 0)$, $(0, -1, 0)$, $(-1, -1, 0)$;

T_2 has vertexes $(0, 0, 0)$, $(-1, -1, 0)$, $(-1, -1, -1)$.

Let $Y = T_1 \cup T_2$ be equipped with the standard Euclidean topology τ and the coordinate wise partial order \preceq . Then

- (i) (Y, τ, \preceq) is a locally compact \preceq -connected lattice (it is not a topological lattice);
- (ii) $\wedge: Y \times Y \rightarrow Y$ is not continuous at point $((0, 0, 0), (0, -1, 0))$. It follows that (Y, τ, \preceq) is not a topological \wedge -semilattice;
- (iii) The canonical map $y \rightarrow y^\downarrow$ from (Y, τ, \preceq) to $(C^\downarrow(Y), \tau_F, \subseteq)$ is not continuous at point $(0, 0, 0)$.

Vietoris and Hausdorff topologies on Hyperspaces of topological \wedge -semilattices

We provide examples of locally compact and order-connected Hausdorff topological \wedge -semilattices (Y, τ, \leq) in which, the order-embedding $x \rightarrow x^\downarrow$

- (i) can topologically order embeds (Y, τ, \leq) in $(C^\downarrow(Y), \tau_F, \subseteq)$;
- (ii) is not continuous at (every) point from (X, τ, \leq) to $(C^\downarrow(X), \tau_V, \subseteq)$;
- (iii) is not continuous at (every) point from (X, τ, \leq) to $(C^\downarrow(X), \tau_H, \subseteq)$.

Example. Define X (a subset of \mathbb{R}^3) by

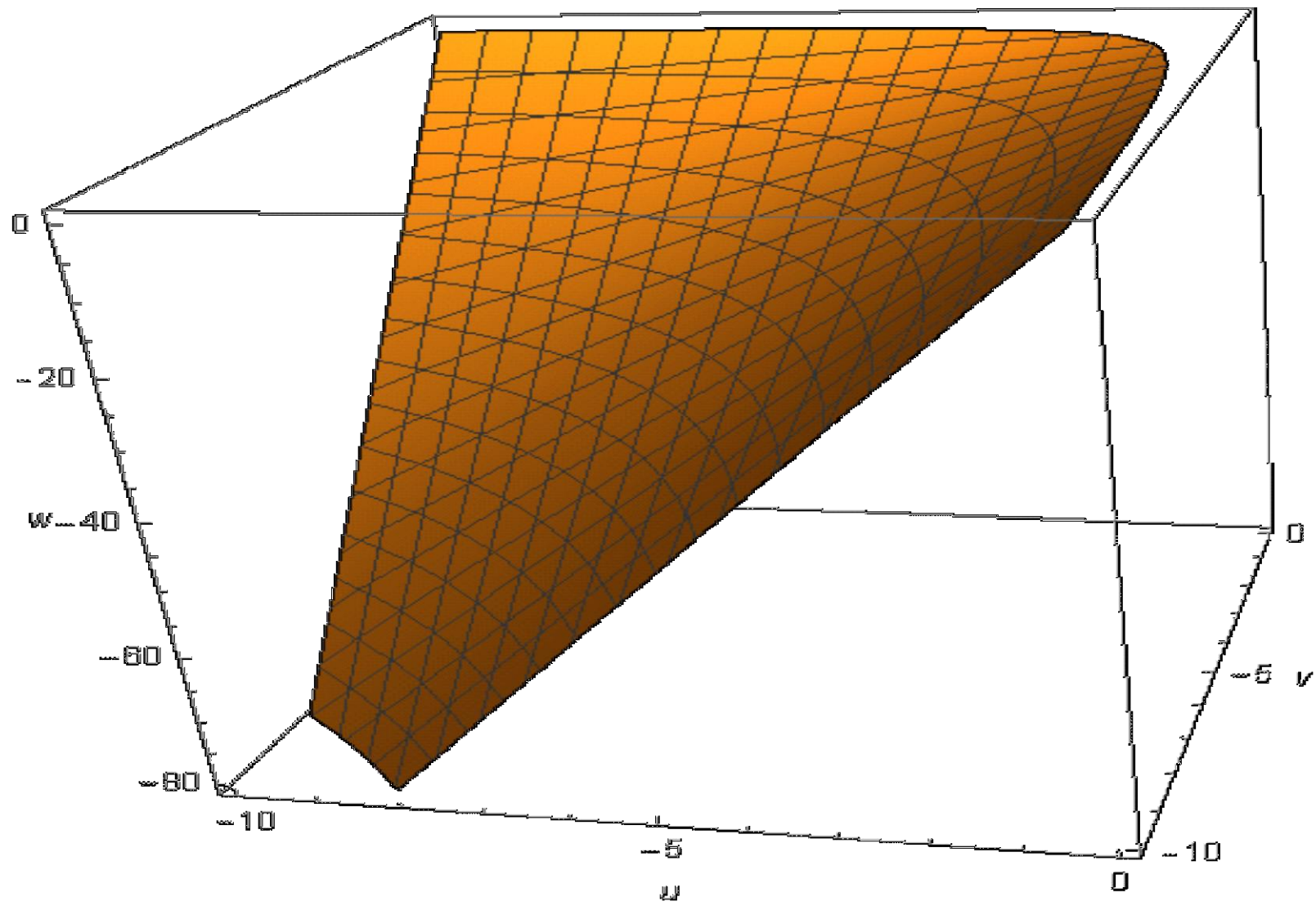
$$X = \{(u, v, w) \in \mathbb{R}^3: -\infty < u, v, w \leq 0 \text{ and } uv + w - 1 \leq 0\}.$$

Let X be equipped with topology τ induced by the standard ordinary Euclidean norm $\|\cdot\|$ and equipped with the coordinate wise partial order \preceq on \mathbb{R}^3 . Then,

(a) (X, τ, \preceq) is a locally compact and order-connected Hausdorff topological \wedge -semilattice.

(b) Moreover, the canonical map $x \rightarrow x^\downarrow$ satisfies that

- (i) It topologically order-embeds (X, τ, \preceq) in $(C(X), \tau_F)$;
- (ii) It is not continuous at every point from (X, τ, \preceq) to $(C^\downarrow(X), \tau_H)$;
- (iii) It is not continuous at every point excepting $(0, 0, 0)$ from (X, τ, \preceq) to $(C^\downarrow(X), \tau_V)$.



Some considerations for further studies

Let (X, τ) be a **Hausdorff** topological space. The Fell topological space $(C(X), \tau_F)$ has the following properties (see G. Beer [1])

- (F₁) $(C(X), \tau_F)$ is compact;
- (F₂) $x \rightarrow \{x\}$ embeds X in $C(X)$ (since τ is Hausdorff, for every $x \in X$, $\{x\} \in C(X)$);
- (F₃) $C_0(X)$ is τ_F -open in $(C(X), \tau_F)$;
- (F₄) When X is first countable, Fell topology convergence of sequence in $C(X)$ is equivalent to their K-P convergence;
- (F₅) When X is uniform space, the Fell topology is coarser than the Hausdorff uniform topology on $C(X)$;

- (F₆) $(C(X), \tau_F)$ is Hausdorff iff X is locally τ -compact; in this setting, as an τ_F -open subset of a τ_F -compact Hausdorff space $C(X)$, $C_0(X)$ is a locally τ_F -compact Hausdorff space;
- (F₇) If X is locally τ -compact and has a countable base W (that is, (X, τ) is second countable), then the following subset of τ_F

$$\{V^- : V \in W\} \cup \{cl(V)^\perp : V \in W \text{ with } V \text{ relatively } \tau\text{-compact}\},$$

forms a countable subbase for τ_F ; that is, it is a countable base of τ_F , which generates the topology τ_F and τ_F is the smallest topology containing it.

In this setting, $(C(X), \tau_F)$ is also second countable), by Urysohn metrization theorem, $(C(X), \tau_F)$ is metrizable. Furthermore, in this setting, it follows that both (X, τ) and $(C(X), \tau_F)$ all are Fréchet–Urysohn spaces, that is, for every subset S of X , the sequential closure of S is the τ -closure of S in X .

(F₈) If $C_0(X)$ is Hausdorff in the relative Fell topology τ_F on $C(X)$, then (X, τ) is locally compact (Proposition 5.1.2 in [4] by Beer).

(F₉) If (X, τ) is a Hausdorff locally compact and second countable topological space, then,

- (i) From (F₆), $(C(X), \tau_F)$ is Hausdorff and, as an τ_F -open subset of a τ_F -compact Hausdorff space, $C_0(X)$ is a locally τ_F -compact Hausdorff space;
- (ii) From (F₇), $(C(X), \tau_F)$ is also second countable;
- (iii) Both (X, τ) and $(C(X), \tau_F)$ all are Fréchet–Urysohn spaces;
- (iv) For every subset S of X , the sequential closure of S is the τ -closure of S in X ;
- (v) For every subset T of $C(X)$, the sequential closure of T is the τ_F -closure of T in $C(X)$.

In this setting (F₉), for every subset $T \subseteq C(X)$, we use the sequential closure of T to check the τ_F -closure of T . It follows that, from (F₄), the Fell topology closure property of T can be checked by the K-P sequentially convergence in $C(X)$.

Thank you very much for your attention !!