Nonlinear Fractional Differential Equations

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Nonlinear Fractional Differential Equations

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Thank you to the organizers

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Nonlinear Fractional Differential Equations

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Nonlinear Fractional Differential Equations

- Fractional calculus
- Fractional Differential Equations
- Nonlinear Fractional Differential Equations
- Some models and applications

Classical calculus
FPT, VA, etc.
New methods and ideas
Fractional Calculus
Fractional calculus is the study of integrals and derivatives of any order, not only integer. There are several definitions of fractional integral and fractional derivative due to Riemann, Liouville, Weyl, Hilbert, etc.
A physical meaning of the fractional order in fractional derivatives is as index of memory. Moreover, fractional calculus plays an important role in super diffusive and subdiffusive processes, which makes it a useful tool in real-world applications.
In 1695 Leibniz wrote a letter to L’Hôpital: *Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*

“WHAT IF $n = \frac{1}{2}$ IN $\frac{d^n f(x)}{dx^n}$”
It will lead to a paradox, from which one day useful consequences will be drawn
\[ I^1 f(t) = \int_0^t f(s) ds \]

\[ I^2 f(t) = I^1 (I^1 f)(t) = \int_0^t (t - s) f(s) ds \]

\[ I^n f(t) = \frac{1}{(n - 1)!} \int_0^t (t - s)^{n-1} f(s) ds \]

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds \quad \alpha > 0, f \in L^1(0, T) \]
\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \quad \alpha > 0, f \in L^1(0,T) \]

\[ I^\alpha : L^1(0,T) \rightarrow L^1(0,T) \]

\[ \alpha > 0, \beta > 0 : I^\alpha \circ I^\beta = I^{\alpha + \beta} \]

\[ \alpha > 0, \lambda > -1 : I^\alpha t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \alpha + 1)} t^{\lambda + \alpha} \]
\[ D^1(t^n) = nt^{n-1} \]

\[ D^2(t^n) = n(n-1)t^{n-2} = \frac{\Gamma(n+1)}{\Gamma(n-2+1)}t^{n-2} \]

\[ D^\alpha(t^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha} \]

\[ D^{1/2} t^1 = \frac{\Gamma(2)}{\Gamma(1/2+1)} t^{1/2} = \frac{2}{\pi} \sqrt{t} \]

\[ D^{1/2} 1 = \frac{1}{\sqrt{\pi}} t^{-1/2} \neq 0 \]

\[ D^{1/2} t^{1/2} = \Gamma(\frac{1}{2}+1)t^0 = \Gamma(3/2) \]
$0 < \alpha < 1 : D^\alpha f = D^1 I^{1-\alpha} f$

$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds$

Riemann-Liouville fractional derivative

$D^\alpha 1 \neq 0$

$D^\alpha t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda-\alpha}$

$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$

$D^\alpha f = I^{1-\alpha} D^1 f$

$C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds$

Liouville-Caputo fractional derivative

$C D^\alpha 1 = 0$
The (integer order) derivative is local

The fractional derivative is global

It has memory!
Geometric application
Tautochrone curve

A tautochrone or isochrone curve (from Greek prefixes tauto- meaning same or iso- equal, and chrono time) is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point.
Tautochrone curve

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Abel’s solution:

$$T(y_0) = \int_{y=y_0}^{y=0} dt = \frac{1}{\sqrt{2g}} \int_{0}^{y_0} \frac{1}{\sqrt{y_0 - y}} \frac{ds}{dy} dy$$
\[ D^\alpha u = 0 \]

\[ D^\alpha u = D^1 I^{1-\alpha} u = 0 \Rightarrow I^{1-\alpha} u = c \]

\[ I^\alpha I^{1-\alpha} u = I^\alpha c = \frac{c}{\Gamma(\alpha + 1)} t^\alpha \]

\[ I^1 u = \frac{c}{\Gamma(\alpha + 1)} t^\alpha \]

\[ D^1 I^1 u = u = \frac{c}{\Gamma(\alpha + 1)} \alpha t^{\alpha-1} = \frac{c}{\Gamma(\alpha)} t^{\alpha-1} \]

\[ D^\alpha u = 0 \Leftrightarrow ct^{\alpha-1}, c \in R \]

\[ D^\alpha u = f \Leftrightarrow u(t) = I^\alpha f(t) + ct^{\alpha-1} \]
\[ 0 < \alpha < 1 \]
\[ D^\alpha u = \lambda u, \lambda \neq 0 \]
\[ u(t) = c \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) \]

\[ D^\alpha u = \lambda u + f, \lambda \neq 0 \]
\[ u(t) = c \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t - s)^\alpha) f(s) ds \]
\[ \lim_{t \to 0^+} t^{1-\alpha} u(t) = c \]
\[ u'(t) = ku(t)(1 - u(t)), \quad t \geq 0 \]

\[ u(t) = \frac{u_0}{u_0 + (1 - u_0) \exp(-kt)}, \quad t \geq 0 \]

Fractional logistic ODE
Power series solution of the fractional logistic equation

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\[ D^\alpha v = v(1 - v), \quad 0 < \alpha \leq 1, \]

\[ v(t) = \sum_{n=0}^{\infty} b_n(\alpha) (t^\alpha)^n. \]

\[ b_{n+1}(\alpha) = \frac{\Gamma(n\alpha + 1)}{\Gamma((n+1)\alpha + 1)} \left[ b_n(\alpha) - \sum_{j=0}^{n} b_j(\alpha) b_{n-j}(\alpha) \right], \quad n \geq 0, \]

\[ b_0(\alpha) = v(0) \]
\[ D^\alpha x(t) = x(t) \cdot [1 - x(t)] \]

\[ \frac{x(t) - x^2(t)}{(1 - x(t))^{2/\alpha}} = \frac{x_0 - x_0^2}{(1 - x_0)^{2/\alpha}} \cdot e^t. \]
\[
\frac{x(t) - x^2(t)}{(1 - x(t))^{2/\alpha}} = \frac{x_0 - x_0^2}{(1 - x_0)^{2/\alpha}} \cdot e^t.
\]

- Classical Logistic ODE: \( \alpha = 1 \)
- Caputo-Fabrizio fractional DE: \( \alpha = 0.1 \)
- Caputo fractional DE: \( \alpha = 1/2 \)

\( x(0) = 1/2 \)
Solution of a fractional logistic ordinary differential equation

Juan J. Nieto

Caputo-Fabrizio fractional DE $\alpha = 0.1$
Fractional-Order Logistic Differential Equation with Mittag–Leffler-Type Kernel

\[ D_{\alpha, \beta, \lambda}^\gamma x(t) = x(t)(1 - x(t)) \]
\[ \Lambda(\alpha, \beta, \gamma, \lambda) D^{\gamma}_{\alpha, \beta, \lambda} x(t) = x(t)(1 - x(t)), \]

\[ \Lambda(\alpha, \beta, \gamma, \lambda) = \begin{cases} 
\frac{\lambda}{\lambda - 1} & \alpha = 1, \\
\left( \frac{B(\alpha)}{1 - \alpha} \right)^{(1-a)\gamma \lambda} & \alpha \neq 1.
\end{cases} \]

\[ x(t) = \sum_{n=0}^{\infty} a_n t^{n\xi}. \]

\[ \Lambda(\alpha, \beta, \gamma, \lambda) \sum_{n=0}^{\infty} \frac{c_n}{s^{-\alpha n}} \sum_{n=0}^{\infty} a_n \frac{\Gamma(\xi n + 1)}{s^{\xi n + 1 - \beta + \alpha \gamma}} = \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(\xi n + 1)}{s^{\xi n + 1}} \]
\( \Lambda(\beta, \alpha, \gamma, 0) \mathbb{D}^\gamma_{\beta, \alpha, 0} = C \mathbb{D}^\alpha \)  
(Liouville–Caputo),

\( \Lambda(\beta, \alpha, 0, \lambda) \mathbb{D}^0_{\beta, \alpha, \lambda} = C \mathbb{D}^\alpha \)  
(Liouville–Caputo),

\( \Lambda(\alpha, 0, -1, \alpha/(\alpha - 1)) \mathbb{D}^{-1}_{\alpha, 0, \alpha/(\alpha - 1)} = AB \mathbb{D}^\alpha \)  
(Atangana–Baleanu),

\( \Lambda(1, 0, -1, \alpha/(\alpha - 1)) \mathbb{D}^{-1}_{\alpha, 0, \alpha/(\alpha - 1)} = CF \mathbb{D}^\alpha \)  
(Caputo–Fabrizio).
Article
Fractional-Order Logistic Differential Equation with Mittag–Leffler-Type Kernel

Iván Area* and Juan J. Nieto**

Logistic function solution with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Caputo–Fabrizio logistic differential Equation (29) in $[0, 2]$ for $\alpha = 0.75$, in orange. From left to right and top to bottom the approximations are shown for $n = 3, n = 5, n = 7$,..
Future directions
Open Problems

- Control of fractional systems
- Sobolev spaces of fractional order
- Fractional logistic equation
- Fractional Laplacian
- General M-L functions
- Fractional models
- Fractional Navier-Stokes equations
- Fractional epidemic models
- Digital Twins
- COVID-19
Control

Given $x_0$ (initial state) $\rightarrow x_f$ (final state)
Find $u$ such that $x(0) = x_0$, $x(T) = x_f$

$x'(t) = Ax(t) + Bu(t)$

$K = (B | AB | A^2B | \ldots | A^{n-1}B)$
\[ x'(t) = Ax(t) + Bu(t) \]

\[ K = (B|AB|A^2B|\ldots|A^{n-1}B) \]

Theorem 2. The fractional system (7) is controllable if and only if the Kalman matrix \( K \) has full rank.

Basic Control Theory for Linear Fractional Differential Equations With Constant Coefficients

Sebastián Buedo-Fernández\(^1,2\) and Juan J. Nieto\(^1,2\)
Fractional model of COVID-19 applied to Galicia, Spain and Portugal

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\[
\begin{align*}
D^\alpha S(t) &= -\beta \frac{I}{N} S - l \beta \frac{H}{N} S - \beta \frac{P}{N} S, \\
D^\alpha E(t) &= \beta \frac{I}{N} S + l \beta \frac{H}{N} S + \beta \frac{P}{N} S - \kappa E, \\
D^\alpha I(t) &= \kappa \rho_1 E - (\gamma_a + \gamma_i) I - \delta_i I, \\
D^\alpha P(t) &= \kappa \rho_2 E - (\gamma_a + \gamma_i) P - \delta_p P, \\
D^\alpha A(t) &= \kappa (1 - \rho_1 - \rho_2) E, \\
D^\alpha H(t) &= \gamma_a (I + P) - \gamma_r H - \delta_h H, \\
D^\alpha R(t) &= \gamma_i (I + P) + \gamma_r H, \\
D^\alpha F(t) &= \delta_i I(t) + \delta_p P(t) + \delta_h H(t),
\end{align*}
\]
On a new and generalized fractional model for a real cholera outbreak

\[
\begin{align*}
\Lambda_{t}^{-1} \mathcal{D}_{t}^{q} S(t) &= \Lambda - (\psi + \mu)S \\
\Lambda_{t}^{-1} \mathcal{D}_{t}^{q} I(t) &= - (\alpha_{1} + \mu + \gamma) \\
\Lambda_{t}^{-1} \mathcal{D}_{t}^{q} Q(t) &= - (\alpha_{2} + \mu + \epsilon) \\
\Lambda_{t}^{-1} \mathcal{D}_{t}^{q} R(t) &= - (\mu + \phi_{1})R(t) \\
\Lambda_{t}^{-1} \mathcal{D}_{t}^{q} V(t) &= - (\phi_{2} + \mu)V(t) \\
\Lambda_{t}^{-1} \mathcal{D}_{t}^{q} C(t) &= - \sigma C(t) + \theta I(t)
\end{align*}
\]
A Digital Twin of a Compartmental Epidemiological Model based on a Stieltjes Differential Equation

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Abstract

We introduce a digital twin of the classical compartmental SIR (Susceptible, Infected, Recovered) epidemic model and study the interrelation between the digital twin and the system. In doing so, we use Stieltjes derivatives to feed the data from the real system to the virtual model which, in return, improves it in real time. As a byproduct of the model, we present a precise mathematical definition of solution to the problem. We also analyze the existence and uniqueness of solutions, introduce the concept of Main Digital Twin and present some numerical simulations with real data of the COVID-19 epidemic, showing the accuracy of the proposed ideas.
Figure 8: Digital twin (DT) solution to the multivalued problem (infectious individuals) in continuous lines. In discontinuous line the real data.
Digital Twin for epidemic of COVID-19

Video at

https://twitter.com/CITMAga/status/1511267398405570562
Thank you for your attention

Nonlinear Fractional Differential Equations

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