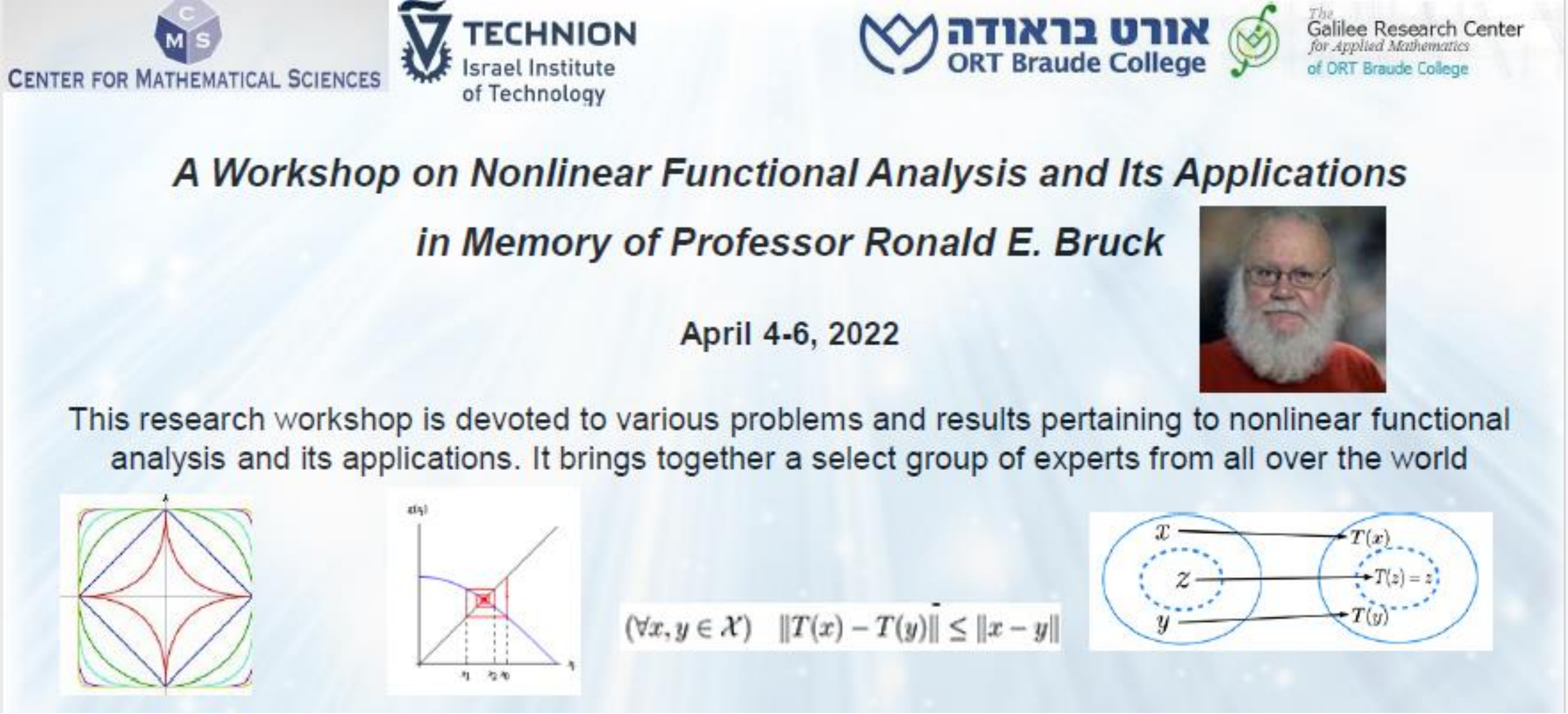

Nonlinear Fractional Differential Equations

Juan J. Nieto

USC

April 5th, 2022

USC Dornsife

Aprox 9000 km

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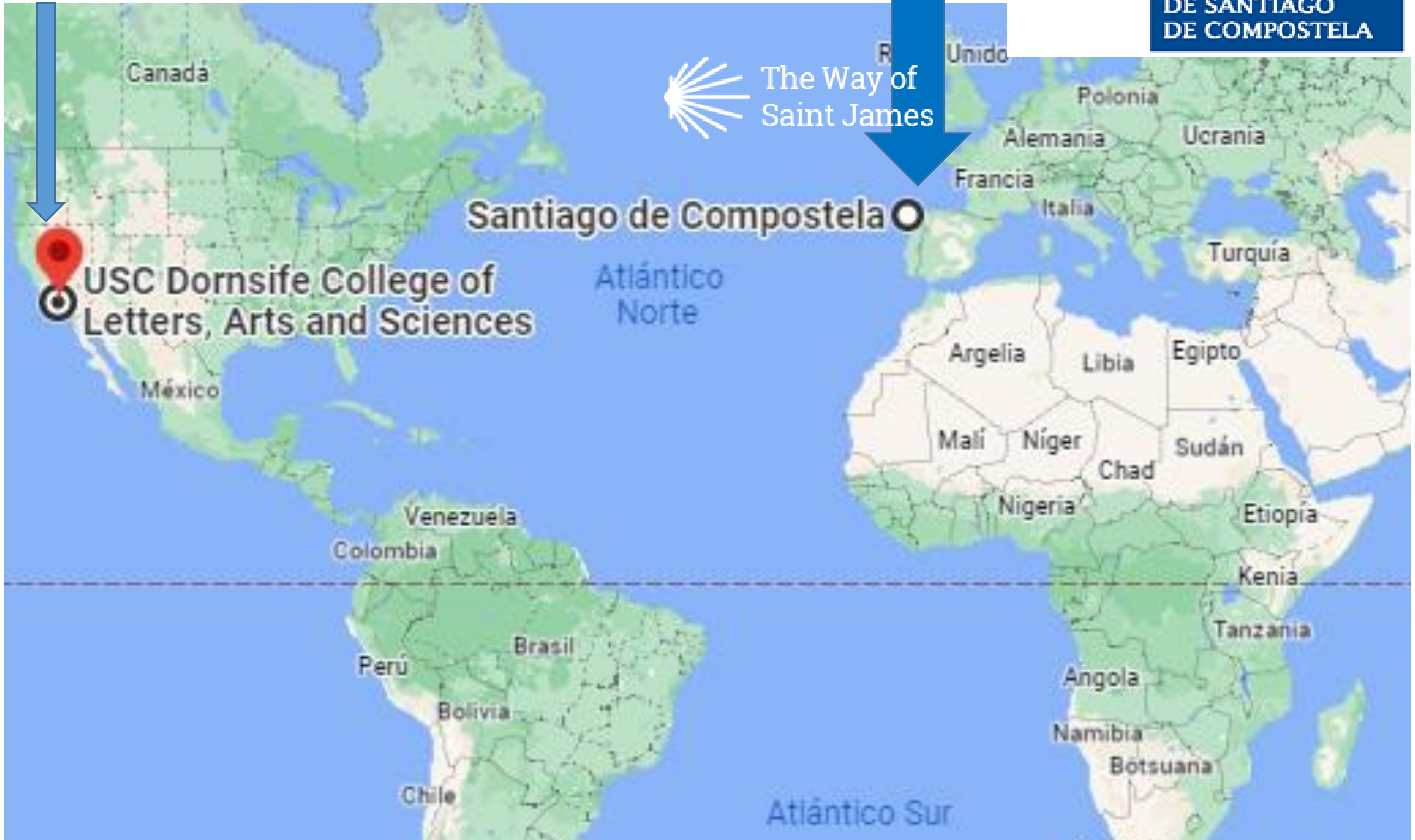

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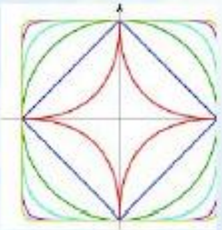
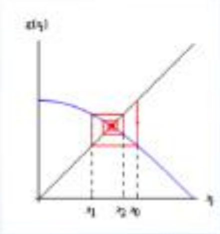
A Workshop on Nonlinear Functional Analysis and Its Applications

in Memory of Professor Ronald E. Bruck

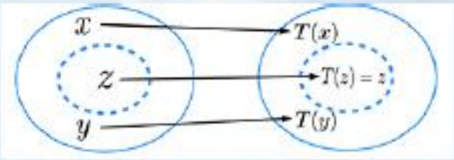
April 4-6, 2022

This research workshop is devoted to various problems and results pertaining to nonlinear functional analysis and its applications. It brings together a select group of experts from all over the world





$$(\forall x, y \in \mathcal{X}) \quad \|T(x) - T(y)\| \leq \|x - y\|$$



Nonlinear Fractional Differential Equations

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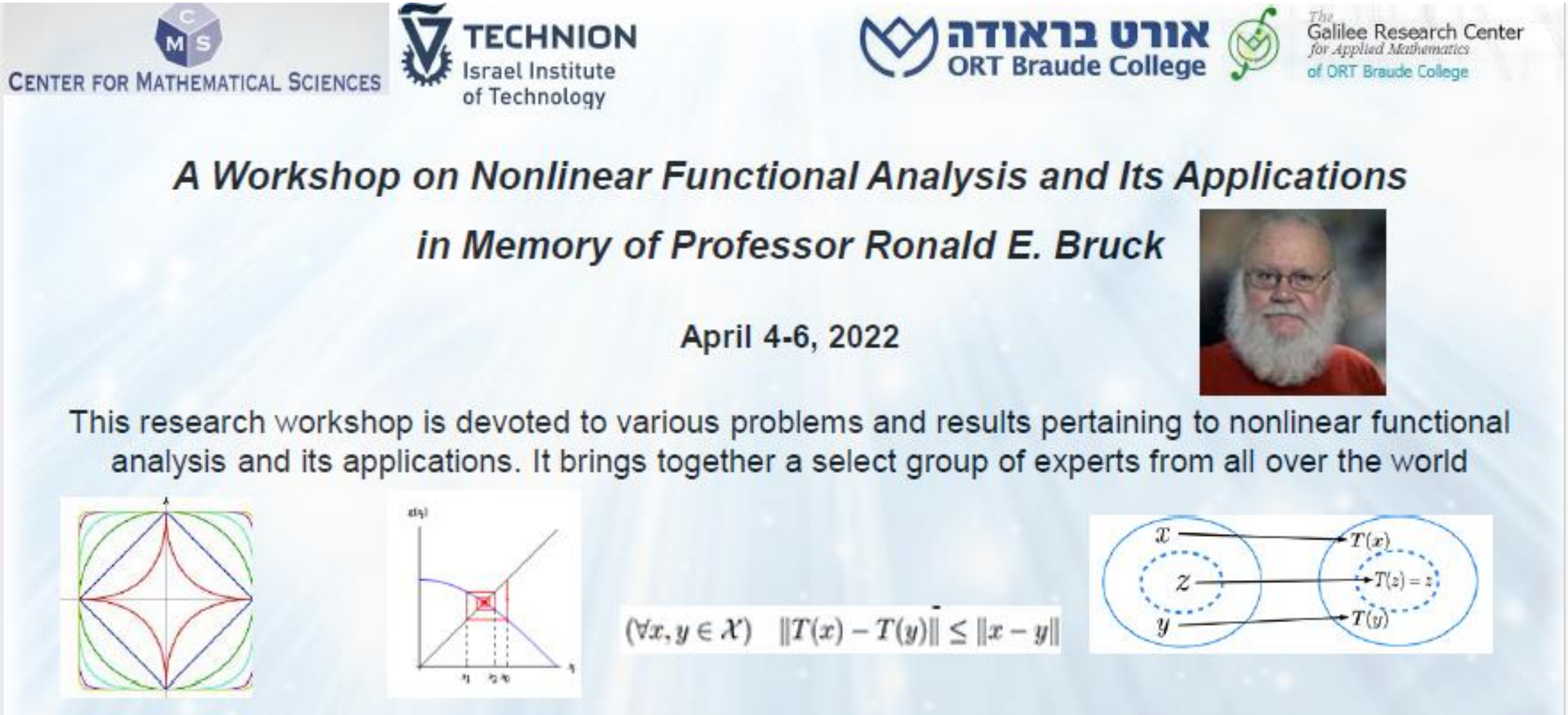
Juan J. Nieto

CITMAga, USC

April 5th, 2022



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Department of Statistics, Mathematical Analysis and Optimization

Nonlinear Fractional Differential Equations

Juan J. Nieto

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Nonlinear Fractional Differential Equations

- Fractional calculus — Classical calculus
- Fractional Differential Equations
- Nonlinear Fractional Differential Equations — FPT, VA, etc.
- Some models and applications — New methods and ideas

Fractional Calculus

Fractional calculus is the study of integrals and derivatives of any order, not only integer. There are several definitions of fractional integral and fractional derivative due to Riemann, Liouville, Weyl, Hilbert, etc.

A physical meaning of the fractional order in fractional derivatives is as index of memory. Moreover, fractional calculus plays an important role in super diffusive and sub diffusive processes, which makes it a useful tool in real world applications.

In 1695 Leibniz wrote a letter to L'Hôpital: *Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*

“ WHAT IF $n = \frac{1}{2}$ IN $\frac{d^n f(x)}{dx^n}$ ”

It will lead to a paradox, from which one day useful consequences will be drawn

$$I^1 f(t) = \int_0^t f(s)ds$$

$$I^2 f(t) = I^1(I^1 f)(t) = \int_0^t (t-s) f(s)ds$$

$$I^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)ds$$

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \qquad \alpha > 0, f \in L^1(0,T)$$

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \qquad \alpha > 0, f \in L^1(0,T)$$

$$I^\alpha : L^1(0,T) \to L^1(0,T)$$

$$\alpha > 0, \beta > 0 : I^\alpha \circ I^\beta = I^{\alpha+\beta}$$

$$\alpha > 0, \lambda > -1 : I^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha}$$

$$D^1(t^n) = nt^{n-1}$$

$$D^2(t^n) = n(n-1)t^{n-2} = \frac{\Gamma(n+1)}{\Gamma(n-2+1)}t^{n-2}$$

$$D^\alpha(t^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}$$

$$D^{1/2}t^1 = \frac{\Gamma(2)}{\Gamma(\frac{1}{2}+1)}t^{1/2} = \frac{2}{\pi}\sqrt{t}$$

$$D^{1/2}1 = \frac{1}{\sqrt{\pi}}t^{-1/2} \neq 0$$

$$D^{1/2}t^{1/2} = \Gamma(\frac{1}{2}+1)t^0 = \Gamma(3/2)$$

$$0 < \alpha < 1 : D^\alpha f = D^1 I^{1-\alpha} f$$

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds$$

Riemann-Liouville fractional derivative

$$D^\alpha 1 \neq 0$$

$$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$$

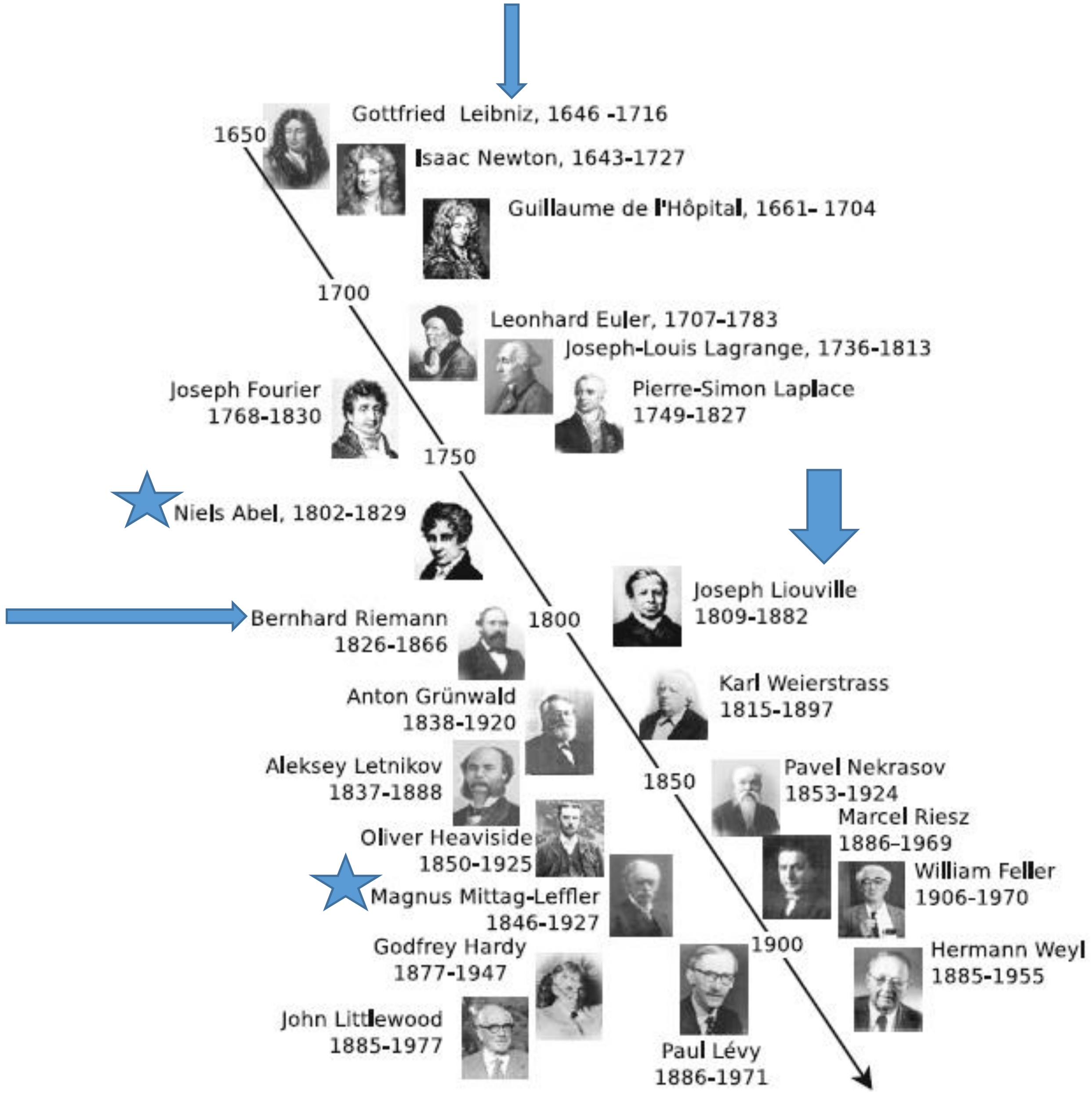
$$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$$

$$D^\alpha f = I^{1-\alpha} D^1 f$$

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds$$

Liouville-Caputo fractional derivative

$${}^C D^\alpha 1 = 0$$



Gottfried Leibniz, 1646 -1716
1650
Isaac Newton, 1643-1727
Guillaume de l'Hôpital, 1661- 1704
1700
Leonhard Euler, 1707-1783
Joseph-Louis Lagrange, 1736-1813
Joseph Fourier
1768-1830
Pierre-Simon Laplace
1749-1827
1750
Niels Abel, 1802-1829
Joseph Liouville
1809-1882
Bernhard Riemann
1826-1866
1800
Karl Weierstrass
1815-1897
Anton Grünwald
1838-1920
Aleksey Letnikov
1837-1888
1850
Pavel Nekrasov
1853-1924
Marcel Riesz
1886–1969
Oliver Heaviside
1850-1925
William Feller
1906-1970
Magnus Mittag-Leffler
1846-1927
Godfrey Hardy
1877-1947
1900
Hermann Weyl
1885-1955
John Littlewood
1885-1977
Paul Lévy
1886-1971

The (integer order) derivative is local

The fractional derivative is global

It has memory!

Geometric application

Tautochrone curve

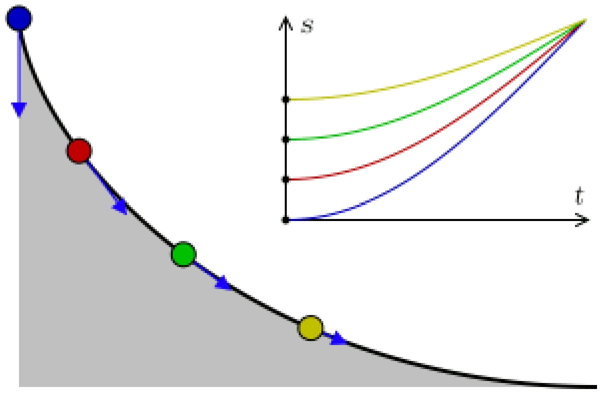
A **tautochrone** or isochrone curve (*from Greek prefixes tauto- meaning same or iso- equal, and chrono time*) is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point.

Tautochrone curve

A tautochrone or isochrone curve (from Greek prefixes tauto- meaning same or iso- equal, and chrono time) is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point.

Abel's solution:

$$T(y_0) = \int_{y=y_0}^{y=0} dt = \frac{1}{\sqrt{2g}} \int_0^{y_0} \frac{1}{\sqrt{y_0 - y}} \frac{ds}{dy} dy$$



$$D^\alpha u = 0$$

$$D^\alpha u = D^1 I^{1-\alpha} u = 0 \Rightarrow I^{1-\alpha} u = c$$

$$I^\alpha I^{1-\alpha} u = I^\alpha c = \frac{c}{\Gamma(\alpha+1)} t^\alpha$$

$$I^1 u = \frac{c}{\Gamma(\alpha+1)} t^\alpha$$

$$D^1 I^1 u = u = \frac{c}{\Gamma(\alpha+1)} \alpha t^{\alpha-1} = \frac{c}{\Gamma(\alpha)} t^{\alpha-1}$$

$$D^\alpha u = 0 \Leftrightarrow c t^{\alpha-1}, c \in R$$

$$D^\alpha u = f \Leftrightarrow u(t) = I^\alpha f(t) + c t^{\alpha-1}$$

$$0 < \alpha < 1$$

$D^\alpha u = \lambda u, \lambda \neq 0$

$$u(t) = c\,\Gamma(\alpha)\,t^{\alpha-1}\,E_{\alpha,\alpha}(\lambda t^\alpha)$$

$D^\alpha u = \lambda u + f, \lambda \neq 0$

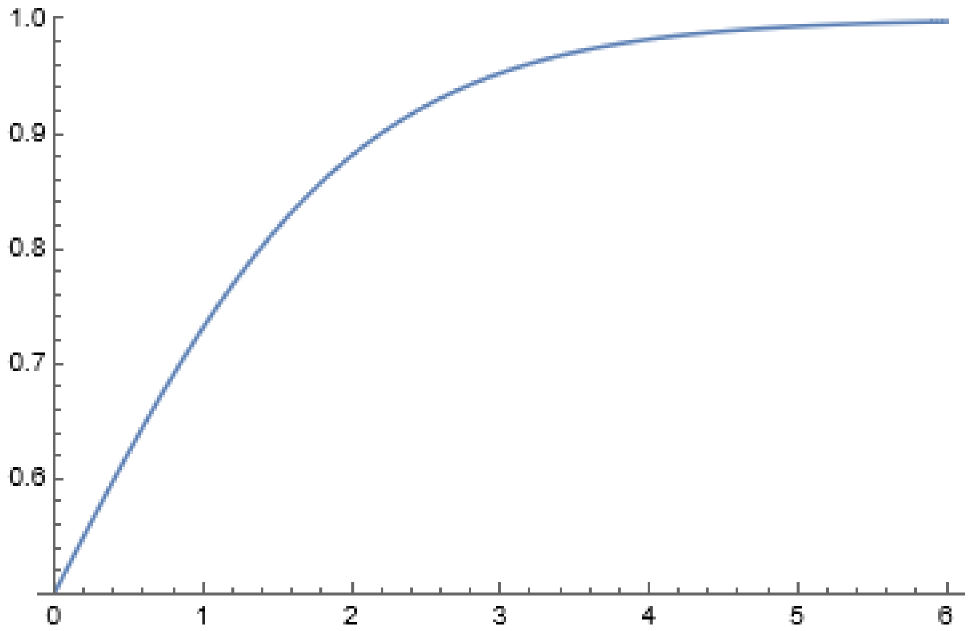
$$u(t) = c\,\Gamma(\alpha)\,t^{\alpha-1}\,E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) f(s) ds$$

$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = c$$

$$u'(t) = ku(t)(1 - u(t)), \quad t \geq 0$$

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)\exp(-kt)}, \quad t \geq 0$$

Fractional logsitic ODE



Physica A 573 (2021) 125947

Contents lists available at ScienceDirect

Physica A

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Power series solution of the fractional logistic equation

I. Area [a,1,*], J.J. Nieto [b,1]

[a] Universidade de Vigo, Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo, Campus As Lagoas-Ourense, 32004 Ourense, Spain
[b] Instituto de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

$$D^{\alpha}v = v(1-v), \quad 0 < \alpha \leq 1,$$

$$v(t) = \sum_{n=0}^{\infty} b_n(\alpha)(t^{\alpha})^n.$$

$$b_{n+1}(\alpha) = \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}\left[b_n(\alpha) - \sum_{j=0}^{n} b_j(\alpha)b_{n-j}(\alpha)\right], \quad n \geq 0,$$

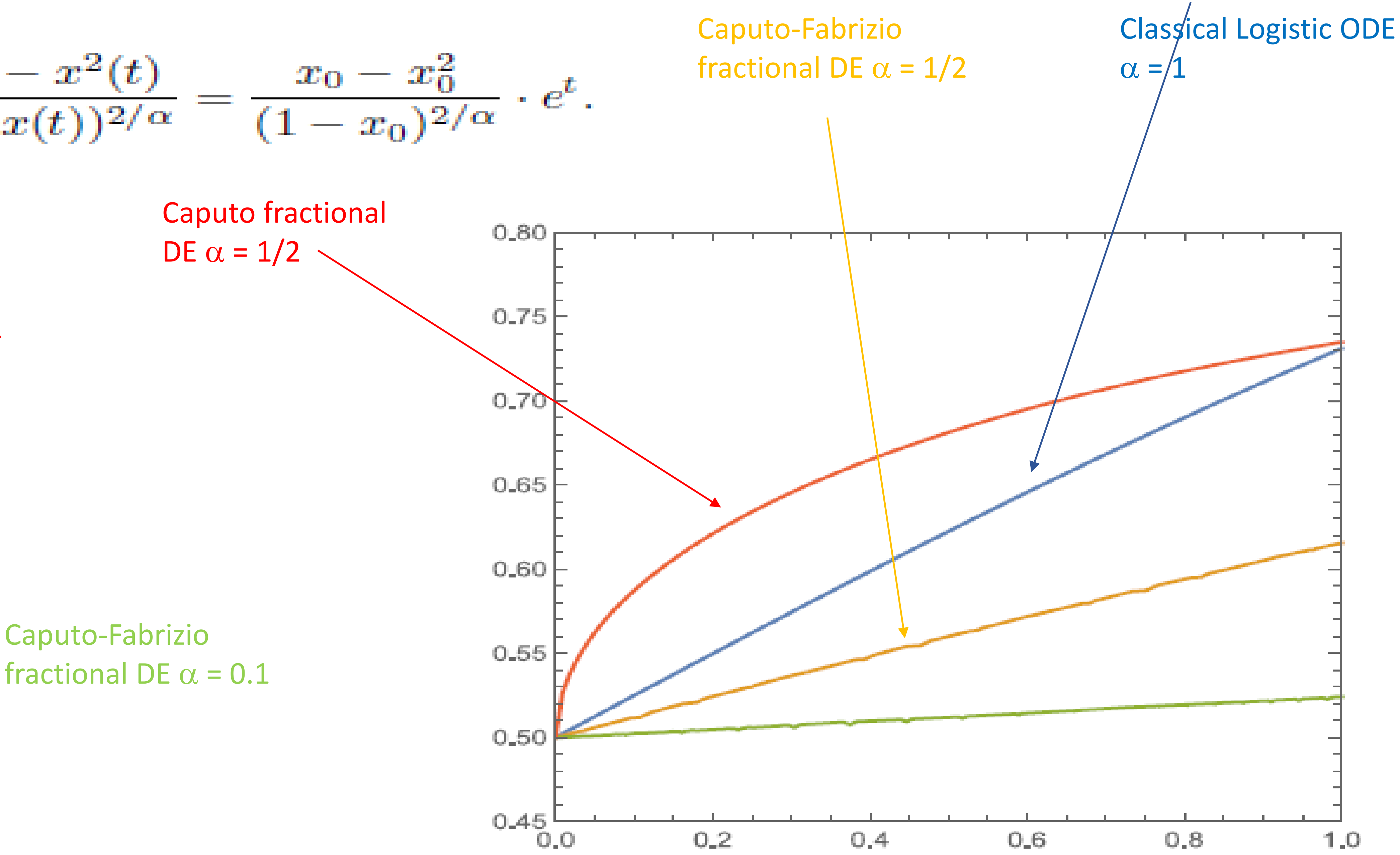
$$b_0(\alpha) = v(0)$$

$$\mathcal{D}^{\alpha} x(t) = x(t) \cdot [1 - x(t)]$$

$$\frac{x(t) - x^2(t)}{(1 - x(t))^{2/\alpha}} = \frac{x_0 - x_0^2}{(1 - x_0)^{2/\alpha}} \cdot e^t.$$

$$\frac{x(t) - x^2(t)}{(1 - x(t))^{2/\alpha}} = \frac{x_0 - x_0^2}{(1 - x_0)^{2/\alpha}} \cdot e^t .$$

x(0)=1/2



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Applied Mathematics Letters 123 (2022) 107568

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Applied Mathematics Letters

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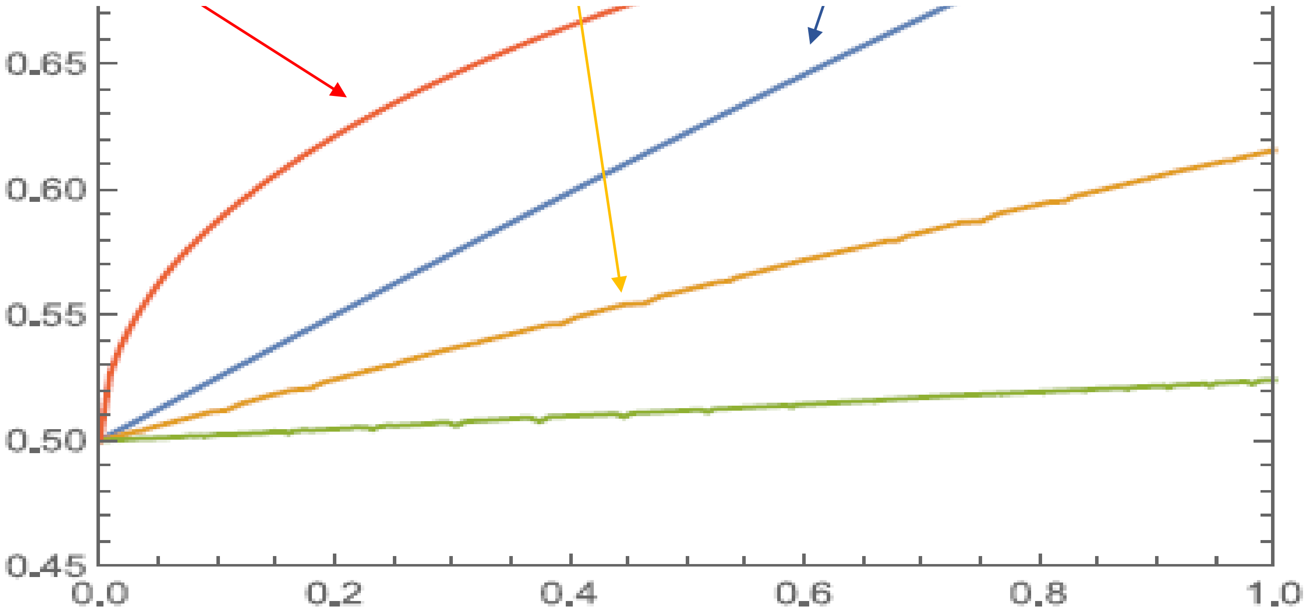
Solution of a fractional logistic ordinary differential equation

Juan J. Nieto

Caputo-Fabrizio
fractional DE $\alpha = 0.1$



Fractional-Order Logistic Differential Equation with Mittag–Leffler-Type Kernel

$$\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda} x(t) = x(t)(1 - x(t))$$

$$\Lambda(\alpha,\beta,\gamma,\lambda)\mathbb{D}^{\gamma}_{\alpha,\beta,\lambda}x(t) = x(t)(1 - x(t)),$$

$$\Lambda(\alpha,\beta,\gamma,\lambda) = \begin{cases} 1 - \dfrac{\lambda}{\lambda - 1}, & \alpha = 1, \\ \left(\dfrac{B(\alpha)}{1-\alpha}\right)^{\frac{(1-\alpha)\gamma\lambda}{\alpha}}, & \alpha \neq 1. \end{cases}$$

$$x(t) = \sum_{n=0}^{\infty} a_n t^{n\xi}.$$

$$\Lambda(\alpha,\beta,\gamma,\lambda)\sum_{n=0}^{\infty}\frac{c_n}{s^{-\alpha n}}\sum_{n=0}^{\infty}a_n\frac{\Gamma(\xi n+1)}{s^{\xi n+1-\beta+\alpha\gamma}} = \sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(\xi n+1)}{s^{\xi n+1}}$$

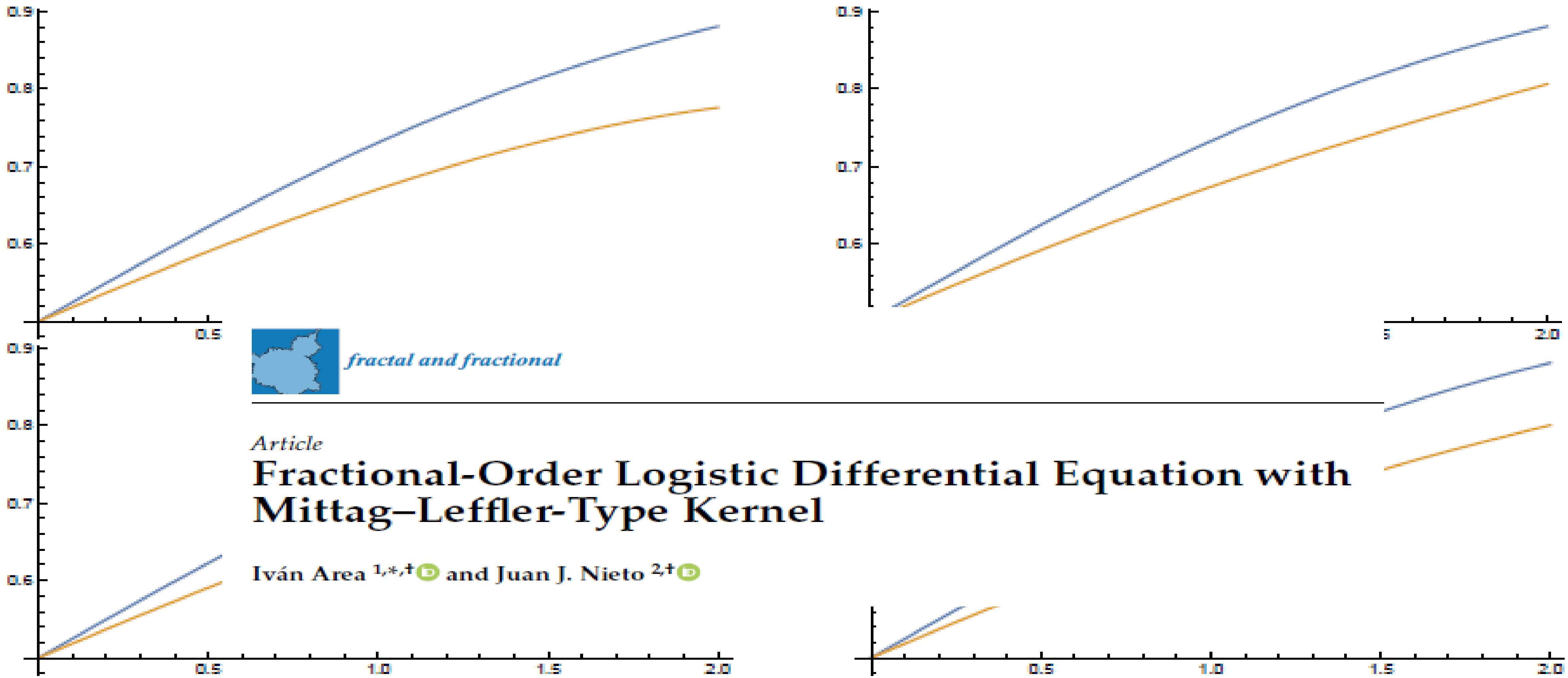
$$
\begin{aligned}
\Lambda(\beta,\alpha,\gamma,0)\mathbb{D}^{\gamma}_{\beta,\alpha,0} &= {}^{\mathrm{C}}\mathbb{D}^{\alpha} && \text{(Liouville–Caputo)},\\
\Lambda(\beta,\alpha,0,\lambda)\mathbb{D}^{0}_{\beta,\alpha,\lambda} &= {}^{\mathrm{C}}\mathbb{D}^{\alpha} && \text{(Liouville–Caputo)},\\
\Lambda(\alpha,0,-1,\alpha/(\alpha-1))\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)} &= {}^{\mathrm{AB}}\mathbb{D}^{\alpha} && \text{(Atangana–Baleanu)},\\
\Lambda(1,0,-1,\alpha/(\alpha-1))\mathbb{D}^{-1}_{\alpha,0,\alpha/(\alpha-1)} &= {}^{\mathrm{CF}}\mathbb{D}^{\alpha} && \text{(Caputo–Fabrizio)}.
\end{aligned}
$$

fractal and fractional

Article

Fractional-Order Logistic Differential Equation with Mittag–Leffler-Type Kernel

Iván Area [1,*,†] and Juan J. Nieto [2,†]



Logistic function solution with $x(0) = 1/2$, in blue, as well as some approximations of the solution to the Caputo–Fabrizio logistic differential Equation (29) in $[0, 2]$ for $\alpha = 0.75$, in orange. From left to right and top to bottom the approximations are shown for $n = 3$, $n = 5$, $n = 7$,

Future directions

Open Problems

Control of fractional systems

Sobolev spaces of fractional order

Fractional logistic equation

Fractional Laplacian

General M-L functions

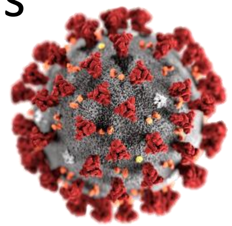
Fractional models

Fractional Navier-Stokes equations

Fractional epidemic models

Digital Twins

COVID-19



$$x'(t) = Ax(t) + Bu(t)$$

$$K = (B|AB|A^2B|\ldots|A^{n-1}B)$$

Control

Control

Given x_0 (initial state) → x_f (final state)

Find u such that $x(0) = x_0$, $x(T) = x_f$

$$x'(t) = Ax(t) + Bu(t)$$

$$K = (B|AB|A^2B|\ldots|A^{n-1}B)$$

-

Theorem 2. *The fractional system (7) is controllable if and only if the Kalman matrix K has full rank.*

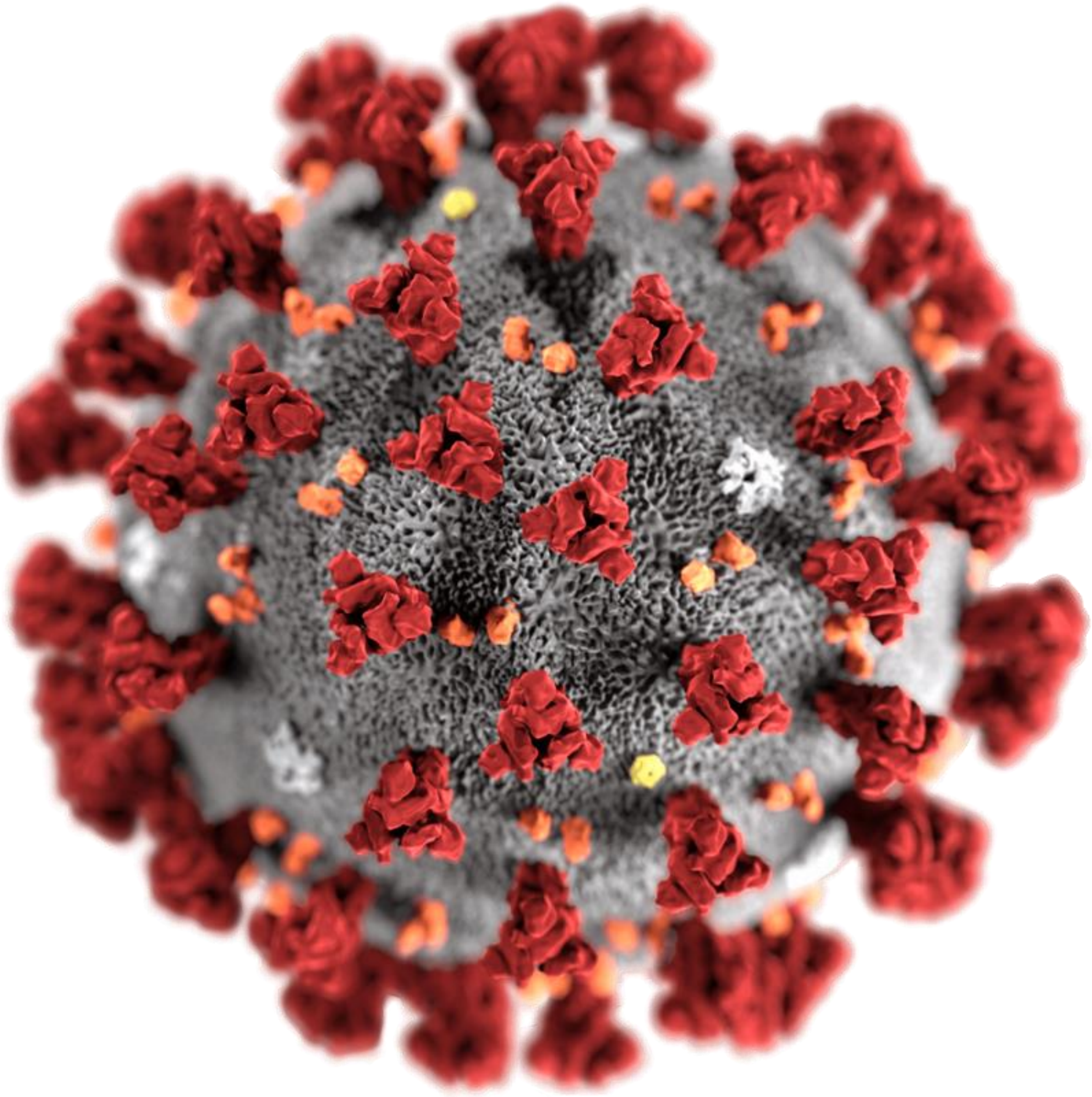
frontiers in Physics

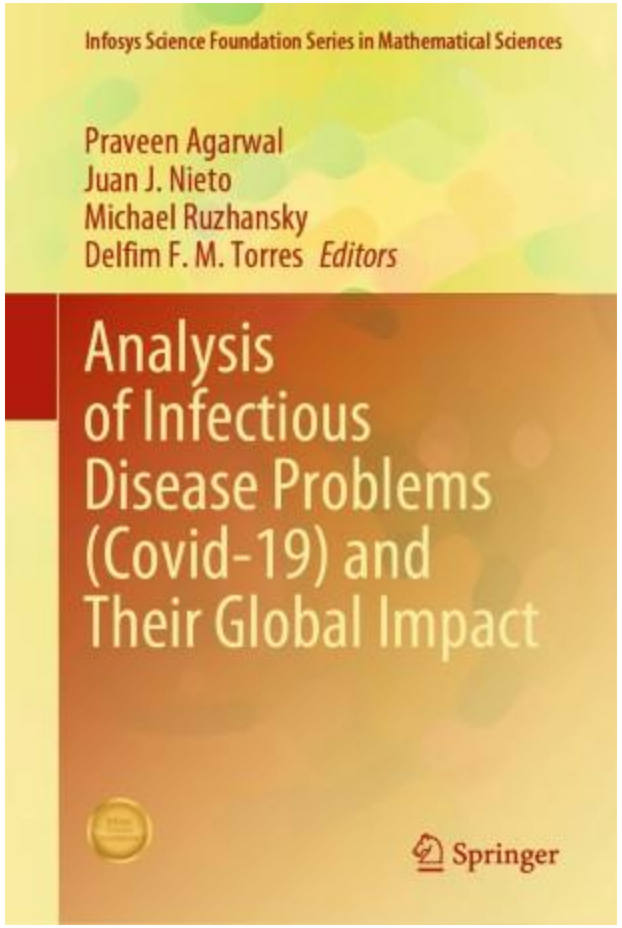
ORIGINAL RESEARCH
published: 24 September 2020
doi: 10.3389/fphy.2020.00377

Check for updates

Basic Control Theory for Linear Fractional Differential Equations With Constant Coefficients

Sebastián Buedo-Fernández [1,2†] and Juan J. Nieto [1,2†]*

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Michael Ruzhansky
Delfim F. M. Torres *Editors*
Analysis of Infectious Disease Problems (Covid-19) and Their Global Impact
Springer

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Fondo COVID-19

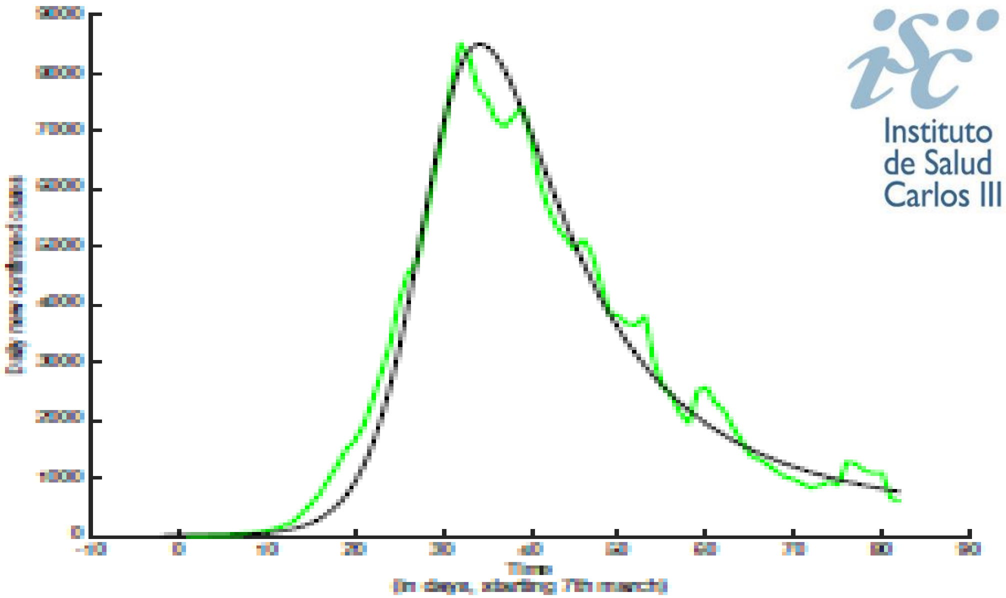
Instituto de Salud Carlos III

Fractional model of COVID-19 applied to Galicia, Spain and Portugal

Faïçal Ndaïrou[a,b], Iván Area[b], Juan J. Nieto[c], Cristiana J. Silva[a], Delfim F.M. Torres[a,*]

[a] Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
[b] Universidade de Vigo, Departamento de Matemática Aplicada II, E. E. Aeronáutica e do Espazo, Campus de Ourense, 32004 Ourense, Spain
[c] Instituto de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela 15782, Spain

$$
\begin{cases}
{}^{C}D^{\alpha}S(t) = -\beta\frac{I}{N}S - l\beta\frac{H}{N}S - \beta'\frac{P}{N}S, \\
{}^{C}D^{\alpha}E(t) = \beta\frac{I}{N}S + l\beta\frac{H}{N}S + \beta'\frac{P}{N}S - \kappa E, \\
{}^{C}D^{\alpha}I(t) = \kappa\rho_1 E - (\gamma_a + \gamma_i)I - \delta_i I, \\
{}^{C}D^{\alpha}P(t) = \kappa\rho_2 E - (\gamma_a + \gamma_i)P - \delta_p P, \\
{}^{C}D^{\alpha}A(t) = \kappa(1 - \rho_1 - \rho_2)E, \\
{}^{C}D^{\alpha}H(t) = \gamma_a(I + P) - \gamma_r H - \delta_h H, \\
{}^{C}D^{\alpha}R(t) = \gamma_i(I + P) + \gamma_r H, \\
{}^{C}D^{\alpha}F(t) = \delta_i I(t) + \delta_p P(t) + \delta_h H(t),
\end{cases}
$$



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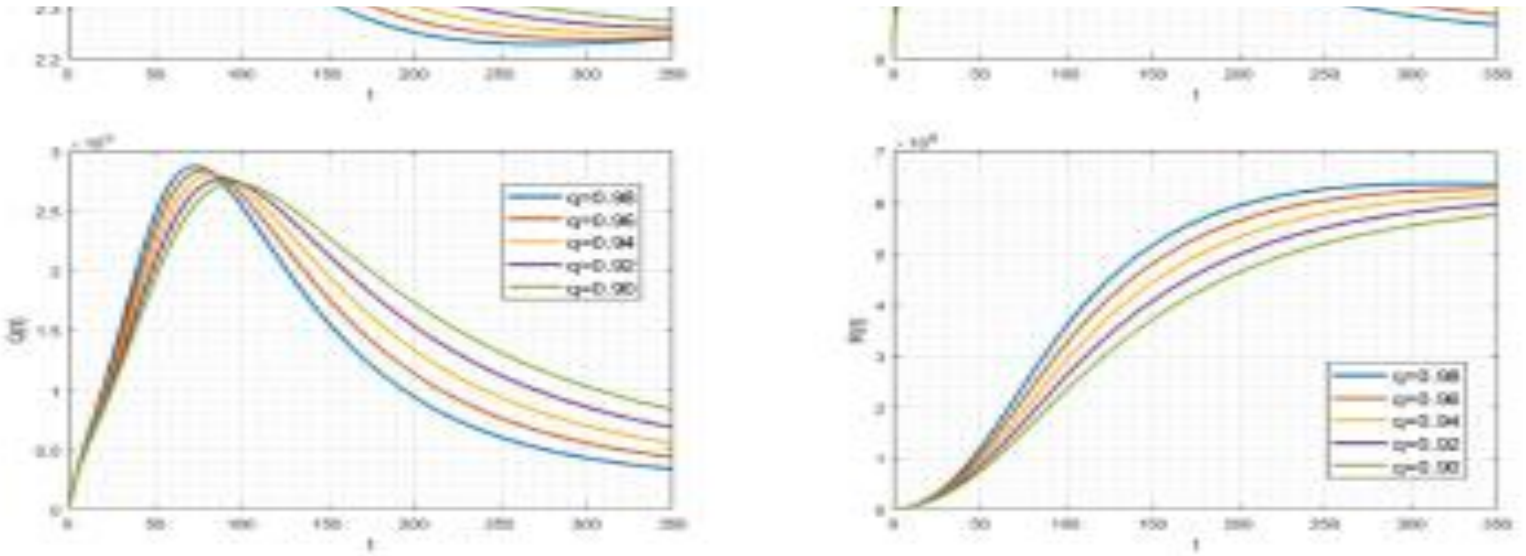
On a new and generalized fractional model for a real cholera outbreak

$$\begin{cases} \lambda^{q-1}{}_0^C\mathscr{D}_t^q S(t) = \Lambda - (\psi + \mu)S \\ \lambda^{q-1}{}_0^C\mathscr{D}_t^q I(t) = -(\alpha_1 + \mu + \gamma) \\ \lambda^{q-1}{}_0^C\mathscr{D}_t^q Q(t) = -(\alpha_2 + \mu + \epsilon \\ \lambda^{q-1}{}_0^C\mathscr{D}_t^q R(t) = -(\mu + \varphi_1)R(\\ \lambda^{q-1}{}_0^C\mathscr{D}_t^q V(t) = -(\varphi_2 + \mu)V(\\ \lambda^{q-1}{}_0^C\mathscr{D}_t^q C(t) = -\sigma C(t) + \theta I(\end{cases}$$



Mathematical Methods in the Applied Sciences

A Digital Twin of a Compartmental Epidemiological Model based on a Stieltjes Differential Equation

Iván Area[a], F.J. Fernández[b], Juan J. Nieto[b,*], F. Adrián F. Tojo[b]

[a] *Universidade de Vigo, Departamento de Matemática Aplicada II, E. E. Aeronáutica e do Espazo, Campus de Ourense, 32004 Ourense, Spain*
[b] *Instituto de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain*

Abstract

We introduce a digital twin of the classical compartmental SIR (Susceptible, Infected, Recovered) epidemic model and study the interrelation between the digital twin and the system. In doing so, we use Stieltjes derivatives to feed the data from the real system to the virtual model which, in return, improves it in real time. As a byproduct of the model, we present a precise mathematical definition of solution to the problem. We also analyze the existence and uniqueness of solutions, introduce the concept of Main Digital Twin and present some numerical simulations with real data of the COVID-19 epidemic, showing the accuracy of the proposed ideas.

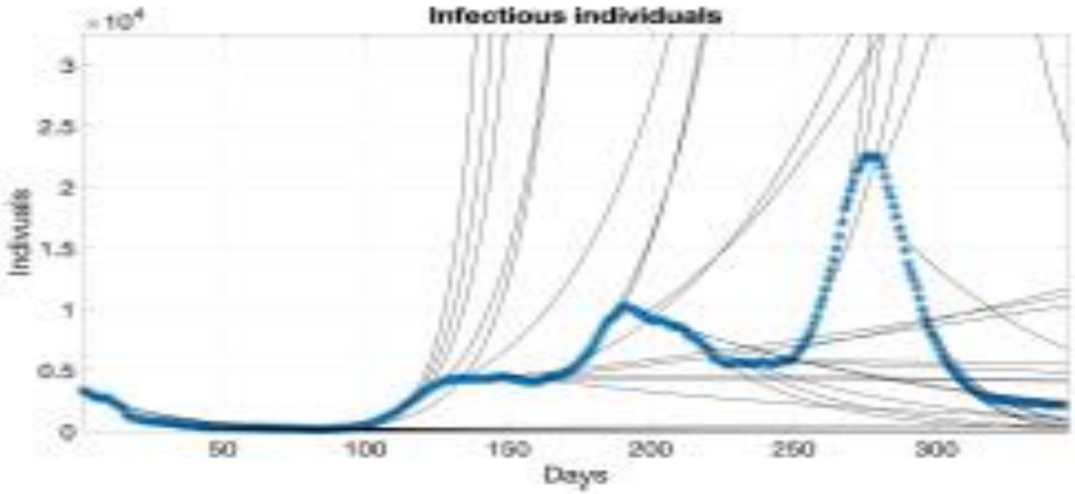


Figure 8: Digital twin (DT) solution to the multivalued problem (infectious individuals) in continuous lines. In discontinuous line the real data

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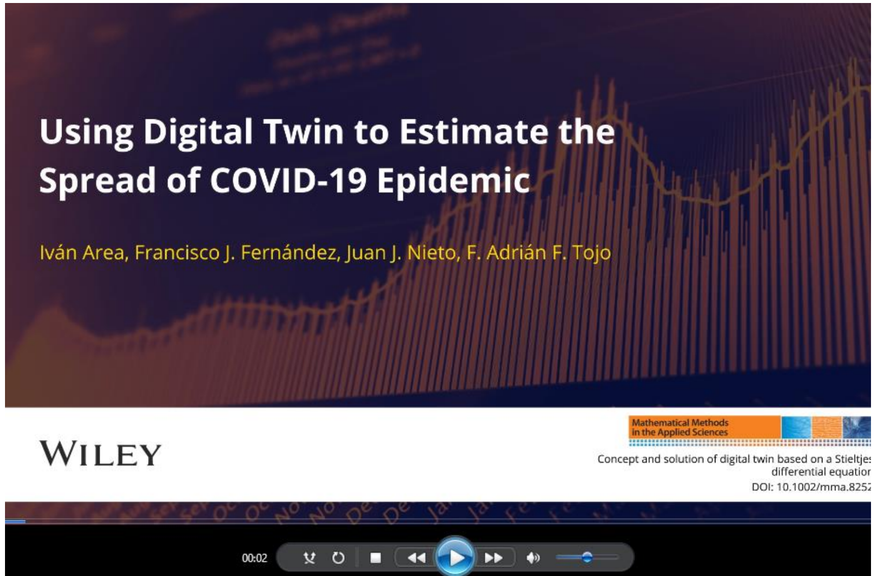
Digital Twin for epidemic of COVID-19

Video at

https://twitter.com/CITMAga/status/1511267398405570562



Thank you for your attention

Nonlinear Fractional Differential Equations

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