Residuality Properties of Certain Classes of Convex Functions on Normed Linear Spaces

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A joint work with Simeon Reich and Alexander J. Zaslavski

A workshop on Nonlinear Functional Analysis and Its Applications in memory of Professor Ronald E. Bruck

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Main ideas and tools

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Definition

A subset Z of a topological space Y is called *residual* if it contains a countable intersection of open and dense subsets of Y.

Theorem (Baire,1899)

Let X be a complete pseudo-metric space. Then the intersection of a countable family of open and dense subsets of X is itself dense in X.

The spaces under consideration

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The uniform topology τ on $\mathfrak{M}$ is determined by the basis for uniformity

$$E(n) = \left\{ (f, g) \in \mathfrak{M} \times \mathfrak{M} : |f(x) - g(x)| < n^{-1} \ \forall x \in K \cap B(n) \right\}.$$

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$\mathfrak{M}_l$, $\mathfrak{M}_c$ and $\mathfrak{M}_b$ are closed subsets of $\mathfrak{M}$ (completely metrizable space) with respect to this uniform topology.

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For each $\tilde{x} = (x, t) \in X \times \mathbb{R}$ and each nonempty set $A \subset X \times \mathbb{R}$, the distance from $\tilde{x}$ to A is

$$\rho\,(\tilde{x}, A) := \inf_{(a,s)\in A}\{\|(x, t) - (a, s)\|_\infty\}\,.$$

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$$d_{AW}(f, g) := \sum_{n=1}^{\infty} 2^{-n} \min \left\{ 1, \sup_{\tilde{x} \in B_{\|\cdot\|_\infty}(n)} \left| \rho(\tilde{x}, \operatorname{epi}(f)) - \rho(\tilde{x}, \operatorname{epi}(g)) \right| \right\}.$$

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It is the same as the uniform topology determined by the basis for uniformity

$$F(n) = \Big\{(f,g) \in \mathfrak{M}_l \times \mathfrak{M}_l : |\rho(\tilde{x}, \mathrm{epi}(f)) - \rho(\tilde{x}, \mathrm{epi}(g))| < n^{-1} \text{ for each } \tilde{x} \in B_{\|\cdot\|_\infty}(n)\Big\}.$$

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Theorem (Alexandrov and Hausdorff, 1924)

A metrizable space X is completely metrizable if and only if it is a G_δ subset of a complete metric space.

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Example ($\mathfrak{M}_l$ is not complete with respect to the metric d_{AW})

Define a sequence $\{f_n\}_{n=1}^{\infty}$ by $f_n(x) = -n$ for each $x \in K$ and each $n = 1, 2, \ldots$. This is a Cauchy sequence which does not converge in $(\mathfrak{M}_l, d_{AW})$.

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Classes of convex functions under consideration

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- *locally uniformly convex* if for each sequence $\{x_n\}_{n=1}^{\infty} \subset K$ and each $x \in K$,

$$\lambda f(x_n) + (1-\lambda) f(x) - f(\lambda x_n + (1-\lambda) x) \underset{n\to\infty}{\to} 0$$

implies $\|x - x_n\| \underset{n\to\infty}{\to} 0$ for each $0 < \lambda < 1$.

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- *locally uniformly convex* if $\|x - x_n\| \underset{n\to\infty}{\to} 0$ whenever $\|x_n + x\| \underset{n\to\infty}{\to} 2$ for each sequence $\{x_n\}_{n=1}^{\infty}$ in the closed unit ball of center zero and each x in this ball.

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- $(X, \|\cdot\|)$ is strictly convex (respectively, locally uniformly convex) if and only if the square of its norm is a strictly convex function (respectively, locally uniformly convex function).
- A locally uniformly convex function is strictly convex. In the case where $K = X$ and the dimension of the vector space X is finite, the converse is also true.

Results obtained in previous studies

Let $\mathcal{F}$ and $\mathcal{G}$ denote, respectively, the set of all strictly convex functions and the set of all locally uniformly convex functions defined on K.

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Theorem (Butnariu, Reich and Zaslavski, 2006)

Assume that there exists a continuous strictly convex function $f_* \in \mathfrak{M}_b$. Then $\mathcal{F}$ is residual in $\mathfrak{M}$ with the τ topology, and the sets $\mathcal{F} \cap \mathfrak{M}_l$ and $\mathcal{F} \cap \mathfrak{M}_c$ are residual in, respectively, $\mathfrak{M}_l$ and $\mathfrak{M}_c$ with their relative strong topologies.

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Theorem (Vanderwerff, 2020)

If $(X, \|\cdot\|)$ is a locally uniformly convex (respectively, a strictly convex) real Banach space and $K = X$, then the set $\mathcal{G}$ (respectively, $\mathcal{F}$) is residual in $\mathfrak{M}_l$ with the weak topology.

Theorem (Barshad, Reich and Zaslavski, 2022)

Suppose there exists a strictly convex function $f_* \in \mathfrak{M}_b$. Then the sets $\mathcal{G}$ and $\mathcal{G} \cap \mathfrak{M}_b$ are residual in, respectively, $\mathfrak{M}$ and $\mathfrak{M}_b$ with the relative τ topology. If, in addition, f_* is lower semi-continuous (respectively, continuous), then the set $\mathcal{G} \cap \mathfrak{M}_l$ (respectively, $\mathcal{G} \cap \mathfrak{M}_c$) is residual in $\mathfrak{M}_l$ (respectively, $\mathfrak{M}_c$) with the relative strong topology.

Our results

Theorem (Barshad, Reich and Zaslavski, 2022)

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Theorem (Barshad, Reich and Zaslavski, 2022)

In the case where $K = X$, the relative weak topology of $\mathfrak{M}_b$ is the same as the relative strong topology of $\mathfrak{M}_b$. As a result, if $\mathcal{F} \cap \mathfrak{M}_b \neq \emptyset$, then the set $\mathcal{G} \cap \mathfrak{M}_b$ (and therefore $\mathcal{F} \cap \mathfrak{M}_b$) is residual in $\mathfrak{M}_b$ with both of these topologies.

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Thank you

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