Variable Lebesgue Spaces $L^{p(t)}(\Omega)$ versus $L_p(\Omega)$ spaces when it comes to Fixed Point Theory

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Metric Fixed Point Theory for Nonexpansive Mappings

Let $(X, \|\cdot\|)$ be a Banach space and C be a subset of X. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C$$

- $(X, \|\cdot\|)$ is said to have the FPP if for every closed convex bounded C and for every nonexpansive mapping $T : C \to C$, there is $x \in C$ with $T(x) = x$.
- $(X, \|\cdot\|)$ is said to have the w-FPP if for every convex weakly compact C and for every nonexpansive mapping $T : C \to C$, there is $x \in C$ with $T(x) = x$.

If X reflexive, both properties are alike.

Long-standing open problems:

Characterize those Banach spaces which have the FPP or the w-FPP (in case of no-reflexivity).

We know that Reflexivity + "some geometric property "imply the FPP

Does every reflexive Banach space fulfil the FPP?

Positive results

Theorem (D. Göhde, F. Browder, 1965)

Let $(X, \|\cdot\|)$ be a uniformly convex Banach space. Then X has the FPP.

Theorem (Kirk, 1965)

Let $(X, \|\cdot\|)$ be a Banach space with weak normal structure. Then X has the w-FPP.

$(X, \|\cdot\|)$ has weak normal structure if for every convex weakly compact subset C with $\mathrm{diam}(C) > 0$, there exists $x_0 \in C$ such that

$$C \subset B(x_0, r) \qquad \text{for some } 0 < r < \mathrm{diam}(C).$$

Negative results: find a counterexample

$(\ell_1, \|\cdot\|_1)$ fails to have the FPP

$$C = \overline{co}(e_n) = \{x := (t_n) : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1\}, \; T : C \to C$$

$$T(t_1, t_2, t_3, \cdots) = (0, t_1, t_2, t_3, \cdots)$$

T is fixed point free and $\|Tx - Ty\|_1 = \|x - y\|_1$.

D.E. Alspach, 1981: $L_1[0,1]$ fails to have the w-FPP

$C = \left\{ f : [0,1] \to [0,1], \int_0^1 f dm = \frac{1}{2} \right\}$ convex, w-compact: $T : C \to C$,

$$Tf(t) = \begin{cases} \min\{2f(2t), 1\} & 0 \leq t \leq \frac{1}{2} \\ \max\{2f(2t-1) - 1, 0\} & \frac{1}{2} < t \leq 1 \end{cases}$$

T is fixed point free in C and $\|Tf - Tg\|_1 = \|f - g\|_1 \; \forall f, g \in C$.

FPP and w-FPP are isometric properties (not isomorphic)

Every Banach space containing an isometric copy of $(L_1[0,1], \|\cdot\|_1)$ fails the w-FPP: $C([0,1], \|\cdot\|_\infty)$, $(\ell_\infty, \|\cdot\|_\infty)$, $(L_\infty(\Omega), \|\cdot\|_\infty)$ fail the w-FPP.

Definition (Hagler, 1972)

A Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy (a.i.c.) of ℓ_1 if there exist $(x_n) \subset X$ and $(\epsilon_n) \subset (0,1)$, $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sum_{n=1}^{\infty}|t_n| \qquad \text{for all } (t_n) \in \ell_1.$$

Theorem (P. Dowling, C. Lennard, 1997)

Every Banach space containing an a.i.c. of ℓ_1 fails the FPP

There exist non-reflexive Banach spaces with the FPP

There exist some renormings of ℓ_1 no containing a.i.c. of ℓ_1.

Theorem (P.K. Lin, 2008)

The sequence space ℓ_1 can be renormed to have the FPP. In fact, given $(\gamma_n) \subset (0,1)$ with $\gamma_n \to 1$, $(\ell_1, |\cdot|)$ has the FPP where

$$|x| = \sup_k \gamma_k \sum_{n=k}^{\infty} |x(n)|$$

for $x = (x(n)) \in \ell_1$.

$$\gamma_1 \|x\|_1 \leq |x| \leq \|x\|_1 \qquad \forall x \in \ell_1.$$

Fixed Point Property in Lebesgue spaces $L_p(\Omega)$:

Let (Ω, Σ, μ) be non-atomic σ-finite measure space. TFAE:

1) $1 < p < +\infty$.
2) $(L^p(\Omega), \|\cdot\|_p)$ has the FPP.
3) $(L^p(\Omega), \|\cdot\|_p)$ has the w-FPP

For the purely atomic case ℓ_p:

- ℓ_p has the FPP $\Leftrightarrow 1 < p < +\infty$
- ℓ_p has the w-FPP $\Leftrightarrow 1 \leq p < +\infty$.

Maurey's result: Reflexive subspaces contained in $L_1(\Omega)$.

Theorem (Maurey, 1981)

If X is a closed subspace of $L_1(\Omega)$ that is reflexive then:

$$(X, \|\cdot\|_1) \text{ has the FPP.}$$

Theorem (P. Dowling, C. Lennard, 1997)

Let X be a nonreflexive closed subspace of $L_1(\Omega)$:

Then $(X, \|\cdot\|_1)$ contains an asymptotically isometric copy of ℓ_1.

If X is a closed subspace of $\subset L_1(\Omega)$, then:

$$(X, \|\cdot\|_1) \text{ has the FPP if and only if } X \text{ is reflexive.}$$

Variable Lebesgue Spaces $L^{p(\cdot)}(\Omega)$

Let (Ω, Σ, μ) be a σ-finite space and

$$p : \Omega \to [1, +\infty] : t \to p(t) \text{ measurable function.}$$

Let $\Omega_{<\infty} = \{t : p(t) < +\infty\}$, $\Omega_{\infty} = \{t \in \Omega : p(t) = +\infty\}$.

$$\rho(f) = \int_{\Omega_{<\infty}} |f(t)|^{p(t)} d\mu + \|f 1_{\Omega_\infty}\|_\infty$$

$$L^{p(\cdot)}(\Omega) := \{f \text{ measurable} : \exists \lambda > 0 : \rho(f/\lambda) < +\infty\}$$

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

$(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space

If $p(t) \equiv p$ then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}) = (L_p(\Omega), \|\cdot\|_p)$.

References related to Variable Lebesgue Spaces

- D. V. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis. Birkhauser: Basel 2013.
- L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka. Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics (2011).
- M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000).
- T. Domínguez-Benavides, M. Japón. *The fixed point property on Variable Lebesgue Spaces.* J. Funct. Anal. 280, (2021), 108896.

Reflexivity in Variable Lebesgue Spaces

Assume that (Ω, Σ, μ) is a **non-atomic** measure, $p : \Omega \to [1, +\infty)$ measurable.

$$p_- := \operatorname*{ess\,inf}_{t\in\Omega} p(t), \qquad p_+ := \operatorname*{ess\,sup}_{t\in\Omega} p(t)$$

Theorem (Lukes, Pick, Pokorny, 2011)

The following are equivalent:

- $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ *is reflexive.*
- $1 < p_- \leq p_+ < +\infty$.
- $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ *is uniformly convex.*

The atomic-case: Nakano sequence spaces

Assume $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. In this case $p = (p_n) \subset [1, +\infty)$, $\rho(x) = \sum_{n=1}^{\infty} |x(n)|^{p_n}$ and

$$\|x\|_{(p_n)} = \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \left(\frac{|x(n)|}{\lambda} \right)^{p_n} \leq 1 \right\}$$

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Theorem

Let $(p_n) \subset [1, +\infty)$. TFAE:

1) $1 < \liminf_n p_n \leq \limsup_n p_n < +\infty$.
2) $(\ell^{p_n}, \|\cdot\|_{(p_n)})$ *is reflexive.*

Analysis of the w-FPP in Variable Lebesgue Spaces

Theorem (A necessary condition: $p_+ < +\infty$)

Let (Ω, Σ, μ) be a σ-finite measure space and let $p : \Omega \to [1, +\infty)$ be a measurable function. Assume that $p_+ = +\infty$.

Then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ contains an isometric copy of $(\ell_\infty, \|\cdot\|_\infty)$.

As a consequence: $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ fails to have the w-FPP.

Examples: $L^{1+x}[0, +\infty)$, $L^{1+\frac{1}{1-x}}[0, 1]$ fail the w-FPP.

Variable Lebesgue Spaces with the w-FPP.

Theorem (A complete characterization of the w-FPP)

Let (Ω, Σ, μ) be a σ-finite measure space and let $p : \Omega \to [1, +\infty)$ be measurable. TFAE:

a) $L^{p(\cdot)}(\Omega)$ *satisfies the w-FPP.*

b) $L^{p(\cdot)}(\Omega)$ *does not contain isometrically* $(L^1[0,1], \|\cdot\|_1)$.

c) $p_+ < +\infty$ *and the set* $\{t \in \Omega : p(t) = 1\}$ *can be split into a purely atomic set and a negligible set.*

We prove that $c) \Rightarrow w-$NS.

The FPP in Variable Lebesgue Spaces

If $L^{p(\cdot)}(\Omega)$ is reflexive $\Rightarrow$ $L^{p(\cdot)}(\Omega)$ has the FPP

Theorem (Maurey's result can be generalized)

Assume that $p_+ < \infty$:
If X is a reflexive subspace of $L^{p(\cdot)}(\Omega)$, then $(X, \|\cdot\|_{p(\cdot)})$ has the FPP.

Is there a converse of Maurey's result in the variable case?

Theorem

Let $p : \Omega \to [1, +\infty]$ be measurable.

$L^{p(\cdot)}(\Omega)$ is not reflexive $\Leftrightarrow$ it contains an isomorphic copy of ℓ_1.

The absense of a.i.c. of ℓ_1 except for the trivial case

Question: If X is a non-reflexive subspace of $L^{p(\cdot)}(\Omega)$, is it possible to find an asymptotically isometric copy of ℓ_1 (as in the $L_1(\Omega)$ case)?

Theorem

Let $L^{p(\cdot)}(\Omega)$ be nonreflexive. Then the following are equivalent:

1) *$L^{p(\cdot)}(\Omega)$ contains an asymptotically isometric copy of ℓ_1.*
2) *$L^{p(\cdot)}(\Omega)$ contains an isometric copy of ℓ_1.*

Question: Are FPP and reflexivity equivalent under the framework of Variable Lebesgue spaces (in the same way that both concepts are equivalent in L_p-spaces)?

FPP and reflexivity are not equivalent under the variable scope

- There exist some nonreflexive Variable Lebesgue Spaces $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ with the FPP.

  (In contrast with the situation in the classical Lebesgue $(L^p(\Omega), \|\cdot\|_p)$ family).

- Consequence: There are some classical nonreflexive Banach spaces that fulfil the FPP (without any renorming procedure)

Near-infinity concentrated norms and the FPP

Definition

Let X B.s. with a Schauder basis and let $\|\cdot\|$ be a norm on X. The norm $\|\cdot\|$ is called near-infinity concentrated (n.i.c.) if:

1. For every $\epsilon > 0$ there exists some $k \in \mathbb{N}$ such that

$$\left[\|x\| + \limsup_n \|x_n\|\right](1-\epsilon) \leq \limsup_n \|x + x_n\|$$

   whenever $k \leq x$ and $(x_n)_n$ is a block basic sequence (b.b.s.).

2. For every $k \in \mathbb{N}$, $\exists\ F_k : (0, +\infty) \to [0, +\infty)$ with
   - $\lim_{\lambda \to 0^+} \frac{F_k(\lambda)}{\lambda} = 0$ and satisfying
   - $\limsup_n \|x_n + \lambda z\| \leq \limsup_n \|x_n\| + F_k(\lambda)\|z\|$

   $\forall$ b.b.s. (x_n) with $\liminf_n \|x_n\| \geq 1$, $\forall \lambda \in (0, +\infty)$, $\forall z \in X$ with $z \leq k$, $\|z\| \leq 1$

- E. Castillo-Santos, P. Dowling, H. Fetter, M. Japón, C. Lennard, B Sims, B. Turett, *Near-infinity concentrated norms and the fixed point property for nonexpansive maps on closed, bounded, convex sets.* J. Funct. Anal. 275 (2018), no. 3, 559-576.

Theorem (2018)

If X has a boundedly complete Schauder basis and $\|\cdot\|$ is near-infinity concentrated, then $(X, \|\cdot\|)$ has the FPP.

Theorem

Let $(p_n) \subset (1, +\infty)$ with $\lim_n p_n = 1$.

Then $(\ell_{p_n}, \|\cdot\|_{p_n})$ has a near-infinity concentrated norm.

Consequently:

The space $(\ell_{p_n}, \|\cdot\|_{p_n})$ is a nonreflexive VLS and it has the FPP.

Some open questions related the FPP in VLS:

1) Consider the purely atomic case and (p_n) a bounded sequence in $(1, +\infty)$ with $\liminf_n p_n = 1$:

   Does $(\ell^{p_n}, \| \cdot \|_{p_n})$ have the FPP?

   (We prove the case when $\lim_n p_n = 1$.)

2) More general:

   Assume that $L^{p(\cdot)}(\Omega)$ is non-reflexive and without an isometric copy of ℓ_1:

   Does $(L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)})$ have the FPP?

Some of the main references

- A. Barrera-Cuevas, M. Japón. *New families of non-reflexive Banach spaces with the fixed point property*. J. Math. Anal. Appl. 425 (2015), no. 1, 349-363.
- Castillo-Santos, F. E.; Dowling, P. N.; Fetter, H.; Japón, M.; Lennard, C. J.; Sims, B.; Turett, B. *Near-infinity concentrated norms and the fixed point property for nonexpansive maps on closed, bounded, convex sets*. J. Funct. Anal. 275 (2018), no. 3, 559-576.
- T. Domínguez-Benavides, M. Japón. *The fixed point property on Variable Lebesgue Spaces*. J. Funct. Anal. 280, (2021), 108896.
- Dowling, P. N.; Lennard, C. J. *Every nonreflexive subspace of* $L_1[0,1]$ *fails the fixed point property*. Proc. Amer. Math. Soc. 125 (1997), no. 2, 443-446.
- C. A. Hernández-Linares, M. A. Japón. *A renorming in some Banach spaces with applications to fixed point theory*. J. Funct. Anal. 258 (2010), 3452-3468.
- P. K. Lin, *There is an equivalent norm on* ℓ_1 *that has the fixed point property*. Nonlinear Anal., 68 (8) (2008), 2303-2308.
- B. Maurey, *Points fixes des contractions de certains faiblement compacts de* L^1, Seminar on Functional Analysis, Ecole Polytech., Palaiseau, 1981.

Thank you very much for all your attention.