Linear Functional Analysis in: Metric Spaces

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Any first course on Linear Functional Analysis[1] will cover the fundamental theorems, then will continue to some major applications. What are these fundamental theorems?

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1. Hahn-Banach Theorem
2. Open Mapping Theorem
3. Closed Graph Theorem
4. Banach-Steinhaus Theorem
5. Riesz's Representation Theorem (in Hilbert spaces)

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It is amazing and rare to consider these theorems in the metric nonlinear setting. For example, is there a study of the Hahn-Banach theorem in metric spaces?

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Hahn-Banach theorem in metric spaces

While investigating an extension of the Hahn-Banach theorem to metric spaces, Aronszajn and Panitchpakdi[1] discovered the concept of hyperconvexity or injectivity. The main result of their investigation is the fact that hyperconvex metric spaces are absolute nonexpansive retracts (ANR).

[1] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405-439.

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Examples of hyperconvex metric spaces are the SNCF metric also known as the Paris metric which is an example of the $\mathbb{R}$-tree metric spaces.

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100th Anniversary

In 1922, Banach[1] published the following result (known as the **Banach Contraction Principle**):

Theorem

Let (M, d) be a complete metric space and $T : M \to M$ a contraction mapping, i.e., there exists $K < 1$ such that

$$d\Big(T(x), T(y)\Big) \leq K\, d(x, y)$$

for all $x, y \in M$. Then T has a unique fixed point ω (i.e., $T(\omega) = \omega$), and for each $x \in M$, we have

$$d\Big(T^n(x), \omega\Big) \leq \frac{K^n}{1-K}\, d\Big(T(x), x\Big),$$

which implies $\lim\limits_{n\to\infty} T^n(x) = \omega$.

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3, 133-181 (1922).

Therefore, the originality of Banach is to extend the known result in the context of functions spaces to the newly introduced abstract metric spaces[1].

[1] M. A. Khamsi, M. Pouzet, *A fixed point theorem for commuting families of relational homomorphisms. Applications to metric spaces, ordered sets and oriented graphs*, Topology and its Applications, 273 (2020) 106970

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Very early on, the relaxing of $K < 1$ was investigated.

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The most significant breakthrough happened little more than 40 years later in 1965 in the case $K = 1$ (in this case, the mappings are known as **nonexpansive**).

[1] M. A. Khamsi, M. Pouzet, *A fixed point theorem for commuting families of relational homomorphisms. Applications to metric spaces, ordered sets and oriented graphs*, Topology and its Applications, 273 (2020) 106970

The case of nonexpansive mappings

The fixed point problem for nonexpansive mappings may be stated as:

Problem

Let (X, d) *be a metric space and* $T : X \to X$ *be a nonexpansive mapping, i.e.*

$$d(T(x), T(y)) \leq d(x, y),$$

for any $x, y \in X$. *When does* T *have a fixed point?*

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for any $x, y \in X$. When does T have a fixed point?

It is easy to see that in general the answer is NO. So we made a small change to this problem to come up with:

Problem

Let $(X, \|.\|)$ be a Banach space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive mapping. When does T have a fixed point in C?

The case of nonexpansive mappings

It is easy to see that it is not enough to have bounded closed convex domains to ensure that a nonexpansive mapping has a fixed point. It was natural to add some compactness assumption. Since nonexpansive mappings are continuous, compactness for the strong topology will reduce the problem to the Brouwer's fixed point theorem. The fixed point problem for nonexpansive mappings evolved to become:

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Let $(X, \|.\|)$ be a Banach space and C a nonempty weakly-compact convex subset of X. Let $T : C \to C$ be a nonexpansive mapping. When does T have a fixed point in C?

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Note that in this problem nonexpansiveness is metric in nature while weak-compactness and convexity are closely related to the linear structure of the underlined space.

The case of nonexpansive mappings: 1965

In 1965, three nice and powerful theorems were discovered which set the ground for the modern metric fixed point theory. The first two are similar discovered by Browder and Göhde:

Theorem

If K is a nonempty bounded closed convex subset of a uniformly convex Banach space $(X, \|.\|)$ and if $T : K \to K$ is nonexpansive, then T has a fixed point. Moreover the set of fixed points of T is a closed convex subset of K.

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Recall that a Banach space $(X, \|.\|)$ is said to be uniformly convex provided for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta$$

whenever $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$.

Uniformly convex metric spaces

May be the first to think of this concept are Sekowski and Stachura[1] working with Goebel who asked them to look into a recent paper of Vigué[2]. This work led to a wonderful book on the subject[3]. These works may be seen as the initiators of what is known as the study of CAT(0) metric spaces by metric fixed point theorists.

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[1]K. Goebel, T. Sekowski, and A. Stachura, *Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball,* Nonlinear Analysis, (**4** (1980), 1011-1021.

[2]J.-P. Vigué, *Points fixes d'applications holomorphes dans un produit fini de boules unités d'espaces de Hilbert*, Ann. Mat. Pura Appl. 137 (1984), 245-256.

[3]K. Goebel, and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol.83, Dekker, New York, 1984.

The case of nonexpansive mappings: 1965

The third result is far more powerful and was discovered by Kirk:

Theorem

Let K be a weakly-compact convex subset of a Banach space $(X, \|.\|)$. Assume that K enjoys the normal structure property. Then any nonexpansive mapping $T : K \to K$ has a fixed point.

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First note that uniformly convex Banach spaces are reflexive, then any bounded closed convex nonempty subset is weakly-compact. Moreover, any bounded closed convex nonempty subset of a uniformly convex Banach space enjoys the normal structure property.

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First note that uniformly convex Banach spaces are reflexive, then any bounded closed convex nonempty subset is weakly-compact. Moreover, any bounded closed convex nonempty subset of a uniformly convex Banach space enjoys the normal structure property.

Therefore Kirk's fixed point result is more general than Browder and Göhde's fixed point result.

The case of nonexpansive mappings

The normal structure is metric in nature and was introduced by Brodskii and Milman in 1948. Indeed, let (M, d) be a metric space. Let C be a nonempty bounded subset of M not reduced to one point. C is said to be diametral if and only if

$$R(c) = \sup_{y \in C} d(c, y) = \sup_{x, y \in C} d(x, y) = diam(C),$$

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for any $c \in C$.

Let $(X, \|.\|)$ be a Banach space. Let K be a bounded nonempty closed convex subset of X. K is said to satisfy the normal property if and only if K does not contain any diametral closed convex nonempty subset not reduced to one point. X is said to satisfy the normal property if and only if X does not contain any diametral bounded closed convex nonempty subset not reduced to one point.

The case of nonexpansive mappings

When the Banach space is the Hilbert space H, a stronger normal structure property is satisfied. Indeed, let C be a bounded nonempty closed convex subset of H not reduced to one point. Then there exists $x \in C$ such that

$$R(x) = \sup_{y \in C} \|x - y\| \leq \frac{\sqrt{2}}{2}\, diam(C) < diam(C).$$

This property gave birth to the uniform normal structure property, i.e., a Banach space $(X, \|.\|)$ satisfies the uniform normal structure if there exists $\alpha < 1$ such that for any bounded nonempty closed convex subset C of X, there exists $x \in C$ such that

$$R(x) = \sup_{y \in C} \|x - y\| \leq \alpha\, diam(C).$$

The case of nonexpansive mappings

Most of the research in metric fixed point theory that followed the publications of the three main fixed point theorems focused on the study of the normal structure property in Banach spaces. Very early on, some asked how an extension of Kirk's fixed point theorem to metric spaces will look like?

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The main ingredient to be considered for this extension are:

1. convexity
2. weak-compactness

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Of the two, convexity is may be the easiest to consider. But the weak-compactness is the one that is still eluding mathematicians till now.

Convexity in Metric Spaces

Before we dive into this problem, let us say something about the 1970 extension given by Takahashi[1].

[1] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, I. Kodai Math. Sem. Rep. **22**, No 2, 142–149 (1970).

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Definition

Let (X, d) be a metric space. The function $W : X \times X \times [0, 1] \to X$ defines a convexity structure provided

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y),$$

for any $u, x, y \in X$ and $\lambda \in [0, 1]$. A subset K of X is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

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Note that we have

$$d(x, W(x, y, \lambda)) = (1-\lambda) d(x, y) \ \ and \ \ d(y, W(x, y, \lambda)) = \lambda d(x, y)$$

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Takahashi definition of convexity is more restrictive and mimic the natural linear convexity. In fact, an earlier definition of convexity in metric spaces was introduced by Menger[1]:

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$$d(x, y) = d(x, z) + d(z, y).$$

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Many other definitions of convexity in metric spaces are offered which are closely connected to Menger convexity and the linear convexity. An original approach was used by Penot.

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Convexity in Metric Spaces

In his extension of Kirk's fixed point theorem, Penot[1] used a convexity concept which is set theoretical:

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Definition

Let (X, d) be a metric space. A family $\mathcal{C}$ of subsets of a set X is called an (abstract) convexity structure if

(1) Both $\emptyset$ and X are in $\mathcal{C}$.

(2) $\mathcal{C}$ is stable under intersections; that is, if $\{D_\alpha\}_{\alpha \in I}$ is any nonempty subfamily of $\mathcal{C}$ then $\bigcap_{\alpha \in I} D_\alpha \in \mathcal{C}$.

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Convexity in Metric Spaces

In his work, Penot assumed that convexity structures contain closed balls, i.e., closed balls are convex. Note that the intersection of convexity structures is also a convexity structure. The smallest convexity structure which contains the closed balls is denoted $\mathcal{A}(X)$ and is known as the family of admissible subsets, i.e., intersection of closed balls.

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When X is a Banach space, we have naturally two convexity structures: $\mathcal{C}(X)$ and $\mathcal{A}(X)$ which are the family of closed convex subsets and admissible subsets respectively.

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Convex subsets in the sense of Takahashi form a convexity structure in the sense of Penot. But the converse may not be true in general.

Weak-compactness in metric spaces

Once, we defined convex subsets, we use the fundamental theorem of Smulian[1] which states:

[1] V. Smulian, *On the principle of inclusion in the space of the type (B)*, Mat. Sb. 5 (47) (1939), 327-328. (Russian) MR 1, 335.

Once, we defined convex subsets, we use the fundamental theorem of Smulian[1] which states:

Smulian Characterization

a Banach space X is reflexive if and only if any decreasing sequence of nonempty bounded closed convex subsets of X has a nonempty intersection.

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The normal structure property is metric in nature, therefore both Takahashi and Penot had no issues in defining it for convex bounded subset not reduced to one point.

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Kirk's FPT in metric spaces

Once all the ingredients are in place, we can state the extension of Kirk's fixed point theorem for nonexpansive mappings in metric spaces defined on a convex domain (in the sense of Takahashi or Penot). The interest into the weak-topology in metric spaces stopped right there. So it was more about a mean to get the metric version of Kirk's fixed point theorem.

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In 1988, I looked at this concept (weak-compactness in metric spaces) and asked few questions that find their roots in Banach spaces. This was the first time such approach was considered.

Reflexivity in metric spaces

In 1984, Maluta[1] proved that a Banach space which possesses the uniform normal structure property is reflexive. The proof is highly linear in the sense that Maluta used the properties that characterize the reflexivity in Banach spaces (James characterization of reflexivity).

[1] E. Maluta, *Uniformly normal structure and related coefficients*, Pacific J. Math. vol. Ill ♯2 (1984), 357-369.

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