# An inertial subgradient extragradient method for approximating a solution of an equilibrium problem in an Hadamard manifold

Simeon Reich and Olawale. K. Oyewole

sreich@technion.ac.il

oyewoleok@campus.technion.ac.il

A workshop on Nonlinear Functional Analysis and Its Applications in memory of Professor Ronald E. Bruck.
Technion-Israel institute of technology, Haifa, Israel.

TECHNION Israel Institute of Technology

6th April 2022

# Outline



TECHNION
Israel Institute
of Technology

# Background information

1. Minimax inequality's introduction by Ky Fan [18] in 1972.
2. The term Equilibrium problem by Blum and Oetlli [5], 1994.
3. Further work on the EP given by Muu and Oetlli [27].

Standard examples of the EP.

- Minimization problem.
- Variational inequality.
- Fixed point problem

TECHNION Israel Institute of Technology

# Problem formulation

Let $g : K \times K \to \mathbb{R}$ be a bifunction such that $g(x,x) = 0$ for all $x \in K$, where $K$ is a nonempty subset of a topological space $X$. Then, the EP is to find a point $x \in K$ such that

$$g(x,y) \geq 0 \;\; \forall \; y \in K. \tag{1}$$

We denote by $Sol(g,K)$, the solution set of the EP (1).

TECHNION
Israel Institute
of Technology

# Why Hadamard Manifold?

The study of variational inequality, equilibrium and other related optimization problems are receiving several attentions of researchers in the framework of Riemannian manifolds. Thus, methods and ideas are being extended from linear settings to this more generalized settings.

So why Hadamard manifold?

- convexity
- constraints

TECHNION
Israel Institute
of Technology

# Introduction

- Colao et al. [12] in 2012 introduced the EP to Hadamard manifold.
- Consistency of the problem (using Fan's KKM lemma).

By replacing $M$ with $X$ in (1), we arrive at Colao et al. [12] formulation of the EP.
After the work of Colao et al. [12], there has been

- (2016) Tang et al. [38],
- (2017) Salahuddin [34],
- (2016) Zhou and Huang [41].

# Iterative methods

The development of effective iterative algorithm for approximating the solution of an optimization problem is another interesting direction. Some iterative algorithms for EP

- (1976) Extragradient Method (EGM) by Korpelevich [22]
- (2011) Subgradient Extragradient Method by Censor et al. [9] (SEGM).
- Tseng extragradient method
- Projection extragradient method and so on.

For EPs, Quoc et al. [39] (2008) intoduced an extragradient-like for approximating a solution of a pseudomonotone EP.

TECHNION Israel Institute of Technology

# More contributions

Recent works in this direction

1. (2014) Nguyen et al. [28] introduced a method for finding a common solution of a fixed point and equilibrium problem based on the extragradient method in Quoc et al. [39]
2. (2021) Habib et al. [31] introduced an inertial viscosity subgradient extragradient algorithm for solving the equilibrium problem.

TECHNION Israel Institute of Technology

# Results in Manifold

(2016) Cruz et al. [14] extended the result of Nguyen et al. [28] to the settings of Hadmard manifold by considering the following algorithm: Let $x_1 \in K$ and $\lambda_n > 0$, compute

$$\begin{cases} y_n = \arg\min\limits_{y \in M} \left\{ g(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \\ x_{n+1} = \arg\min\limits_{y \in M} \left\{ g(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \end{cases}$$

such that $0 \leq \lambda_n < \mu < \min\{\frac{1}{c_1}, \frac{1}{c_2}\}$ where $c_1 > 0$ and $c_2 > 0$ are the Lipschitz constants with respect to the bifunction $g$. By replacing $d(x, y)$ with $\|x - y\|$ we obtain the method of Nguyen et al. [28].

# Introduction

Fan et al. [17] method.
Choose $x_1 \in K$ and $\lambda > 0$, $\mu \in (0,1)$. Given the current iterate $x_n \in K$ and $\lambda_n (n \geq 0)$. Calculate $x_{n+1}, \ \lambda_{n+1}$ as

$$\begin{cases} y_n = \arg\min_{y \in K} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \\ x_{n+1} = \arg\min_{y \in K} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\} \end{cases}$$

and

$$\lambda_{n+1} = \left\{ \lambda_n, \frac{\mu[d^2(x_n, y_n) + d^2(x_{n+1}, y_n)]}{2[g(x_n, x_{n+1}) - g(x_n, y_n) - g(y_n, x_{n+1})]_+} \right\}.$$

# Results in Manifold

Ali-Akbari [1] method.
Choose an initial point $x_0 \in K$ and $\lambda$ such that $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. Assume that $x_n \in K$ and we calculate $x_{n+1} \in K$ as follows:

$$\begin{cases} y_n = \arg\min\limits_{y\in K}\left\{f(x_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y)\right\}, \\ x_{n+1} = \arg\min\limits_{y\in T_n}\left\{f(y_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y)\right\} \end{cases}$$

where $T_n = \{y \in M : \langle \exp_{y_n}^{-1} x_n - \lambda v_n, \exp_{y_n}^{-1} y\rangle \leq 0\}$ and $v_n \in \partial_2 g(x_n, y_n)$.

TECHNION
Israel Institute
of Technology

# Motivation

Methods:

(1) Subgradient method of Censor et al. [9].

(2) viscosity approach [2, 15, 20].

(3) Colao et al. [12], Ali-Akbari [1], Fan et al. [17].

(4) Inertial technique [29].

Applications:

1. Competitive exchange economy
2. Product pricing
3. Fractional programming and so on.

# Definitions and Lemmas

Let $M$ be a $m$-dimnesional manifold and $x \in M$, let $T_x M$ be the tangent space of $M$ at $x \in M$. We denote by $TM = \bigcup_{x \in M} T_x M$ the tangent bundle of $M$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called the Riemannian metric on $T_x M$. The corresponding norm to the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x M$ is denoted by $\|\cdot\|_x$.
Given a piecewise smooth curve $\gamma : [a, b] \to M$ joining $x$ to $y$ (i.e $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of $\gamma$ by $l(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Let $\gamma$ be a smooth curve in $M$. A vector field $X$ along $\gamma$ is said to be parallel if $\nabla_{\gamma'} X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector (see [33]).

# Definitions and Lemmas

1. [33] For a complete Riemannian manifold $M$, then the exponential map $\exp_x : T_xM \to M$ at $x \in M$ is defined by $\exp_x v = \gamma_v(1, x), \quad \forall\ v \in T_xM$, where $\gamma_v(\cdot, x)$ is the geodesic starting from $x$ with velocity $v$ (i.e $\gamma_v(0, x) = x$ and $\gamma_v'(0, x) = v$). Then, for any $t$, we have $\exp_x tv = \gamma_v(t, x)$ and $\exp_x \mathbf{0} = \gamma_v(0, x) = x$.
2. The mapping $\exp_x$ is differentiable on $T_xM$ for every $x \in M$. The exponential map has an inverse $\exp_x^{-1} : M \to T_xM$. For any $x, y \in M$, we have $d(x, y) = \| \exp_y^{-1} x\| = \| \exp_x^{-1} y\|$,
3. A complete simply connected Riemannian manifold of nonpositive sectional curvature is said to be an Hadamard manifold. From now, we denote by $M$ a finite dimensional Hadamard manifold.

## Definitions and Lemmas

We now present some useful results and definition which will be useful in the convergence analysis of our main result.

### Proposition

*[33]. Let $x \in M$. The exponential mapping $\exp_x : T_x M \to M$ is a diffeomorphism, for any two points $x, y \in M$, there exists a unique normalized geodesic joining $x$ to $y$, which is expressed by the formula*

$$\gamma(t) = \exp_x t \exp_x^{-1} y, \quad \forall\ t \in [0, 1].$$

TECHNION
Israel Institute
of Technology

# Definitions and Lemmas

The following definitions can be found in [5, 25]. Let $K$ be a nonempty convex subset of $M$. A bifunction $g : M \times M \to \mathbb{R}$ is said to be

(i) monotone on $K$, if

$$\langle g(x,y) + g(y,x) \leq 0, \quad \forall\, x, y \in K;$$

(ii) pseudomontone on $K$, if

$$g(x,y) \geq 0 \Rightarrow g(y,x) \leq 0, \quad \forall\, x, y \in K;$$

(iii) Lipschitz-type continuous, if there exist constants $c_1 > 0$ and $c_2 > 0$, such that

$$g(x,y)+g(y,z) \geq g(x,z)-c_1 d^2(x,y)-c_2 d^2(y,z) \ \forall\, x, y, z \in K.$$

## Definitions and Lemmas

For solving the EP (1), we make the following assumptions for $g$ on $K$ :

(A1) $g$ is pseudomonotone on $K$ and $g(x,x) = 0$ for all $x \in M$;

(A2) $g(\cdot, y)$ is upper semicontinuous for all $y \in M$;

(A3) $g(x, \cdot)$ is convex and subdifferentiable for all fixed $x \in M$;

(A4) $g$ satisfies the Lipschitz-type condition on $M$ i.e
$g(x,y) + g(y,z) \geq g(x,z) - c_1 d^2(x,y) - c_2 d^2(y,z)$.

TECHNION
Israel Institute
of Technology

# Definitions and Lemmas

## Proposition

*Let $M$ be an Hadamard manifold and $x \in M$. Let $\rho_x(y) = \frac{1}{2}d^2(x, y)$, then $\rho_x(y)$ is strictly convex and its gradient at $y$ is given by*

$$\partial\rho_x(y) = -\exp_y^{-1} x.$$

## Proposition

*Let $K$ be a nonempty convex subset of an Hadamard manifold $M$ and $h : K \to \mathbb{R}$ be a convex subdifferentiable and lower semicontinuous on $K$. Then a point $x$ solves the convex minimization problem*

$$\min_{x \in K} h(x) \iff 0 \in \partial h(x) + N_K(x).$$

# Definitions and Lemmas

## Lemma

*[37] Let* $x, y \in K$ *and* $\lambda \in [0, 1]$. *Then, the following properties hold on* $K$.

(i) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$;

(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$;

(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.

TECHNION Israel Institute of Technology

# Definitions and Lemmas

## Lemma

*[32] Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad \forall\ n \geq 1.$$

*If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition*

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

*then $\lim_{n\to\infty} a_n = 0$.*

# Main result

In this section, we propose a strong convergent algorithm for approximating a solution of the EP (1) and then discuss its convergence analysis. The solution set $Sol(g, K)$ is closed and convex [12, 34]. We assume that $Sol(g, K)$ is nonempty.

# Main result

## Algorithm

*Inertial subgradient extragradient method for EP(ISEMEP)*

**Initialization:** *Choose* $x_0, x_1 \in K$, $\lambda_1 > 0$, $\mu \in (0,1)$, *a nonnegative sequence of real numbers* $\{\alpha_n\}$ *such that and* $\sum_{n=1}^{\infty} \alpha_n < +\infty$ *and* $\beta_n \subset (0,1)$ *satisfying*

(C1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$.

**Step 1:** *Given* $x_n, x_{n-1}$ *and* $\lambda_n$. *Compute*

$$\begin{cases} w_n = \gamma_n^0(\theta_n), \\ y_n = \arg\min_{y \in M} \left\{ g(w_n, y) + \frac{1}{2\lambda_n} d^2(w_n, y) \right\}. \end{cases} \tag{2}$$

# Main result

## Algorithm

*If $y_n = w_n$, then stop. Otherwise go to the next step.*

**Step 2:** *Define the half space $T_n$ by*

$$T_n := \{y \in M : \langle \exp_{y_n}^{-1} w_n - \lambda_n v_n, \exp_{y_n}^{y} \rangle \leq 0\}$$

*with $v_n \in \partial_2 g(w_n, y_n)$ and compute*

$$z_n = \arg \min_{y \in T_n} \left\{ g(y_n, y) + \frac{1}{2\lambda_n} d^2(w_n, y) \right\}. \quad (3)$$

**Step 3:** *Compute*

$$x_{n+1} = \gamma_n^1(1 - \beta_n), \quad \forall\ n \geq 0, \quad (4)$$

# Main result

## Algorithm

*where* $\gamma_n^1 : [0,1] \to M$ *is the geodesic joining* $f(x_n)$ *to* $z_n$*, that is* $\gamma_n^1(0) = f(x_n)$ *and* $\gamma_n^1(1) = z_n$ *for all* $n \geq 0$*. Let* $d_n = g(w_n, z_n) - g(w_n, y_n) - g(y_n, z_n)$*. Then,*

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n + \alpha_n, \frac{\mu[d^2(y_n,w_n)+d^2(z_n,y_n)]}{2d_n}\right\}, & d_n > 0, \\ \lambda_n + \alpha_n, & \text{otherwise}. \end{cases} \tag{5}$$

*Set* $n := n+1$ *and return to* **Step 1**.

# Main result

**Remark**

*We observe from Algorithm 6, that the method is self adaptive with the step-size allowed to increase from iteration to iteration unlike the monotone decreasing sequence step-size in [31]. Thus, the dependence of the bifunction $g$ on the Lipschitz constants is dispensed with.*

TECHNION Israel Institute of Technology

# Main result

The set $K$ is a subset of $T_n$. This claim is easily obtained by the Proposition 3, the Normal cone and subdifferential definition.

**Lemma**

*The sequence $\{x_n\}$ defined recursively by Algorithm 6 satisfies the inequality*

$$d^2(z_n, p) \leq d^2(w_n, p) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)[d^2(w_n, y_n) + d^2(y_n, z_n)].$$

TECHNION Israel Institute of Technology

# Main result

In the next result, we show that the sequence $\{x_n\}$ generated by Algorithm 6 has a weak limit.

**Lemma**

*Let $f : K \to K$ be a $\kappa$-contraction mapping. The sequence $\{x_n\}$ generated by Algorithm 6 is bounded.*

We present our main theorem result in the next slide with some highlight of the proof given.

# Main theorem

## Theorem

*Suppose condition (A1)-(A4) and let $f : K \to K$ be a $\kappa$-contraction, then the sequence $\{x_n\}$ generated by Algorithm 6 converges strongly to a point $p = P_{Sol(g,K)}f(p) \in Sol(g,K)$, where $P_{Sol(g,K)}$ is the projection of $K$ onto $Sol(g,K)$.*

TECHNION Israel Institute of Technology

# Application to variational inequalities

Suppose

$$g(x,y) = \begin{cases} \langle Gx, \exp_x^{-1} y\rangle, & \text{if } x, y \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $G : K \to M$ is a mapping, then the equilibrium problem (1) reduces to the variational inequality (VIP):

$$\text{Find} \quad x \in K \quad \text{such that} \quad \langle Gx, \exp_x^{-1} y\rangle \geq 0, \quad \forall\, y \in K. \tag{6}$$

We denote the set of solution of VIP (6) by $VIP(G, K)$.

TECHNION Israel Institute of Technology

# Application to VIP

The mapping $G : K \to M$ is said to be pseudomonotone if

$$\langle Gx, y - x \rangle \geq 0 \Rightarrow \langle Gy, y - x \rangle \geq 0, \quad x, y \in K.$$

Assume that the function $G$ satisfies the following conditions:

(V1) The function $G$ is pseudomonotone on $K$ with $VIP(G, K) \neq \emptyset$.

(V2) $G$ is $L$-Lipschitz continous, that is

$$\|P_{y,x}Gx - Gy\| \leq \|x - y\|, \quad x, y \in K,$$

where $P_{y,x}$ is a parallel transport (see [21]).

(V3) $\limsup_{n\to\infty} \langle Gx_n, \exp_{x_n}^{-1} y \rangle \leq \langle Gp, \exp_p^{-1} y \rangle$ for every $y \in K$ and $\{x_n\} \subset K$ such that $x_n \rightharpoonup p$.

# Application formulation

By replacing the proximal term $\arg\min_{y\in M}\left\{g(x,y)+\frac{1}{2\lambda_n}d^2(x,y)\right\}$ with $P_K(\exp_x(-\lambda_n G(x)))$, where $P_K$ is a projection of $M$ onto $K$ in Algorithm 6, we obtain a method for approximating a point in $VIP(G,K)$. Under this settings we have the following strong convergence theorem for approximating a solution of the VIP (6).

**Theorem**

*Let $f : K \to K$ be a $\kappa$-contraction and $G : K \to M$ be a pseudomonotone operator satisfying conditions $V1$-$V3$, then the sequence $\{x_n\}$ generated by Algorithm 6 converges strongly to an element $p = P_{VIP(G,K)}f(p)$.*

# Application to Convex minimization problem

$$\begin{cases} \min_{x \in K} \phi(x) \end{cases} \tag{7}$$

where $\phi$ is a proper lower semicontinuous convex function of $M$ onto $(-\infty, +\infty]$ such that $M$ is contained in the domain of $\phi$. That is $K \subset dom\phi = \{x \in M : \phi(x) < +\infty\}$. The set of solution of COP (7) is denoted by $COP(\phi, K)$. Suppose we define the bifunction $g : K \times K \to \mathbb{R}$ by $g(x, y) = \phi(y) - \phi(x)$, then $g(x, y)$ satisfies the condition (A1)-(A4) and $COP(\phi, K) = Sol(g, K)$. Let $Prox_{\lambda\phi}$ be the proximal operator of the function $\phi$ of parameter $\lambda > 0$ and $\nabla\phi$ is the gradient of $\phi$. Using the term $Prox_{\lambda\phi}(\exp_x(-\lambda\nabla\phi(x)))$ in place of $\arg\min_{y \in M} \left\{ g(x, y) + \frac{1}{2\lambda_n} d^2(x, y) \right\}$ in Algorithm 6, we obtain a method for minimizing the function $\phi$.

# Numerical example

We consider an extension of the Nash equilibrium model introduced in [16, 21]. In this problem, the bifunction $g : K \times K \to \mathbb{R}$ is given by

$$g(x, y) = \langle Px + Qy + p, y - x \rangle.$$

Let $M$ be give by Space 2 and $K \subset M$ be given by

$$K = \{x = (x_1, x_2, \cdots, x_m) : 1 \leq x_i \leq 100, \quad i = 1, 2, \cdots, m\},$$

$x, y \in K$, $p = (p_1, p_2, \cdots, p_m)^T \in \mathbb{R}^m$ is chosen randomly with elements in $[1, m]$. The matrices $P$ and $Q$ are two square matrices of order $m$ such that $Q$ is symmetric positive semidefinite and $Q - P$ is negative semidefinite. It is known (see [21]) that $g$ is pseudomonotone, satisfies (A2) with Lipschitz constant $c_1 = c_2 = \frac{1}{2}\|Q - P\|$ (see [39, Lemma 6.2]).

# Numerical example cont'd

Table: Computation result for Example **??**.

| | | Algorithm 6 | [17, Algorithm 1] |
|---|---|---|---|
| m=20 | No of Iter. | 23 | 39 |
| | CPU time (sec) | 0.0013 | 2.9229 |
| m=30 | No of Iter. | 23 | 43 |
| | CPU time (sec) | 0.0130 | 3.6771 |
| m=50 | No of Iter. | 41 | 53 |
| | CPU time (sec) | 0.0050 | 5.8712 |
| m=60 | No of Iter. | 35 | 40 |
| | CPU time (sec) | 0.0050 | 5.8712 |

# Advantage of result

The following are some remarks about the current result compared to the existing methods and results in the literature.

(i) The method in this paper uses an adaptive step-size which is allowed to increase from iteration to iteration as against the method in [17] which is monotonically decreasing and the method of [1, 30, 31] which relies on the Lipschitz condition of the bifunction. It is known that the Lipschitz constants can be difficult to estimate which thus affects the efficiency of the method.

(ii) The use of the inertial technique makes the convergence of our faster than the method used in [1, 17].

# Motivation

(iii) With the viscosity method, we obtained a strong convergence theorem which makes our result desirable over the results of Ali-Akbari and Fan et al. [1, 18]. We note that the control parameter of the viscosity step of the method is only required to be non-summable unlike the parameters in [15, 2] which requires an extra condition that the difference between successive parameters be summable. Finally,

(iv) Our result is obtained in the framework of Hadamard manifold unlike the results of [28], [31] and [39] which were obtained in the real Hilbert spaces.

# Conclusion

Thanks for your time.

TECHNION Israel Institute of Technology

M. Ali-Akbari, A subgradient extragradient method for equilibrium problems on Hadamard manifolds, *Int. J. Nonlinear Anal. Appl.*, **13 (1)** (2022), 75–84.

S. Al-Homidan, Q.H. Ansari, F. Babu and J.-C. Yao, Viscosity method with a $f$-contraction mapping for Hierarchical variational inequalities on Hadamard manifolds.

F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.*, **9** (2001), 3–11.

Q.H. Ansari, F. Babu and J.C. Yao, Regularization of proximal point algorithms in Hadamard manifolds. *J. Fixed Point Theory Appl.*, **21(1)**, 25 (2019).

TECHNION
Israel Institute
of Technology

E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63(1–4)** (1994), 123–145.

M.R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences)* vol. 319. Springer, Berlin (1999). https://doi.org/10.1007/978-3-662-12494-9

D.W. Boyd, J.S. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.*, **20** (1969), 335–341.

L.C. Ceng, X. Li, and X. Qin, Parallel proximal point methods for systems of vector optimization problems on Hadamard manifolds without convexity, *Optimization.*, **69(2)** (2020), 357–383.

Y. Censor, A. Gibali and S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert spaces, *J. Optim. Theory Appl.*, **148** (2011), 318-335.

A. Cegielski, *Iterative methods for fixed point problems in Hilbert spaces.* Berlin: Springer; 2012. (Lecture notes in mathematics; 2057).

J. Chen and S. Liu, Extragradient-like method for pseudomontone equilibrium problems on Hadamard manifolds, *J. Ineq. Appl.*, (2020) 2020:205. https://doi.org/10.1186/s13660-020-02473-y.

V. Colao, G. López, G. Marino and V. Martín-Márquez, Equilibrium problems in Hadamard manifolds, *J. Math. Anal. Appl.*, **388 (1)** (2012), 61–77.

- R.W. Cottle and J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space. *J. Optim. Theory Appl.*, **75** (1992), 281–295.
- J.X. Cruz Neto, P.S.M. Santos and P.A. Soares, An extragradient method for equilibrium problems on Hadamard manifolds, it Optim. Lett., **10(6)** (2016), 1327–1336.
- M. Dilshad, A. Khan and M. Akram, Splitting type viscosity methods for inclusion and fixed point problems on Hadamard manifolds, *AIMS Mathematics*, **6 (5)** (2021), 5205–5221.
- F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. II, Springer Series in Operations Research, Springer, New York, 2003.

J. Fan, B. Tan and S. Li, An explicit extragradient algorithm for equilibrium problems on Hadamard manifolds, *Comp. Appl. Math.*, **40: 68** (2021), https://doi.org/10.1007/s40314-021-01427-4

K. Fan, *A minimax inequality and its application*, in: O. Shisha (Ed.), Inequalities, **3**, Academic, New York, (1972), 103–113.

O. P. Ferreira and P. R. Oliveira, Proximal Point Algorithm on Riemannian Manifolds, Optimization: *A Journal of Mathematical Programming and Operations Research.*, **51 (2)** (2002), 257–270.

S. Huang, Approximations with weak contractions in Hadamard manifolds, *Linear Nonlinear Anal.*, **1(2)** (2015), 317–328.

TECHNION
Israel Institute
of Technology

K. Khammahawong, P. Kumam, P. Chaipunya, J.C. Yao, C.F. Wen, and W. Jirakitpuwapat, An extragradient algorithm for strongly pseudomonotone equilibrium problems on Hadamard manifolds. *Thai J. Math.*, **18(1)** (2020), 350–371.

G.M. Korpelevich, An extragradient method for finding saddle points and for other problems, *Ekon. Mat. Metody.*, **12** (1976), 747–56.

C. Li, G. López and V. Martín-Márquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds, *J. Lond. Math. Soc.*, **79 (3)** (2009), 663–683. https://doi.org/10.1112/jlms/jdn087.

C. Li, and J.C. Yao, Variational inequalities for set-valued vector fields on Riemannian manifolds: convexity of the solution set and the proximal point algorithm, *SIAM J. Control Optim.*, **50(4)** (2012), 2486–2514.

G. Mastroeni, On auxiliary principle for equilibrium problems. In: Equilibrium Problems and Variational Models, Erice, 2000. *Nonconvex Optim. Appl.*, Kluwer Academic, Norwell, **68** (2003), 289–298.

A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46–55.

L.D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, *Nonlinear Anal.*, **18** (1992), 1159–1166.

T.T.V. Nguyen, J.J. Strodiot and V.H. Nguyen, Hybrid methods for solving simultaneously an equilibrium problem and

TECHNION Israel Institute of Technology

countably many fixed point problems in a Hilbert space, *J. Optim. Theory Appl.*, **160(3)** (2014), 809–831.

B.T. Polyak, Some methods of speeding up the convergence of iterarive methods *Zh. Vychisl. Mat. Mat. Fiz.*, **4** (1964), 1–17.

H. Rehman, P. Kumam, A. Gibali and W. Kumam, Convergence analysis of a general inertial projection-type method for solving pseudomonotone equilibrium problems with applications, *J. Ineq. Appl.*, **63**, (2021), https://doi.org/10.1186/s13660-021-02591-1

H. Rehman, P. Kumam and K. Sitthithakerngkiet, Viscosity-type method for solving pseudomonotone equilibrium problems in a real Hilbert space with applications, *AIMS Mathematics.*, **6 (2)** (2021), 1538-1560.

TECHNION Israel Institute of Technology

S. Saejung and P. Yotkaew, Approximation of zeros of inverse strongly monotone operator in Banach spaces, *Nonlinear Anal.*, **75** (2012), 742–750.

T. Sakai, *Riemannian geometry. Vol. 149, Translations of mathematical monographs,* Providence (RI): American Mathematical Society; 1996. Translated from the 1992 Japanese original by the author.

S. Salahuddin The existence of solution for equilibrium problems in Hadamard manifolds. *Trans. A. Razmadze Math. Inst.*, **171 (3)** (2017), 381–388.

Y. Shehu, Q.-L. Dong and D. Jiang, Single projection method for pseudo-monotone variational inequality in Hilbert Spaces, *Optimization.*, In press. DOI:10.1080/02331934.2018. 1522636

- G. Stampacchia, *Formes bilineaires coercivites sur les ensembles convexes.* C. R. Acad. Paris. **258** (1964), 4413–4416.
- W. Takahashi, Introduction to Nonlinear and Convex Analysis. Yokohama Publishers, Yokohama (2009).
- G.J. Tang, L.W. Zhou and N.J. Huang, Existence results for a class of hemivariational inequality problems on Hadamard manifolds. Optimization 65(7) (2016), 1451–1461.
- D.Q. Tran, M.L. Dung, and V.H. Nguyen, Extragradient algorithms extended to equilibrium problems. *Optimization.*, **57 (6)** (2008), 749–776.

- J.H Wang, G. López, V. Martín-Márquez, C. Li, Monotone and accretive vector fields on Riemannian manifolds. J. Optim Theory Appl., **146** (2010), 691–708.
- L.-W Zhou and N.-J Huang, Existence of solutions for vector optimization on Hadamard manifolds, *J. Optim. Theory Appl,* **157(1)** (2013), 44–53.