Polynomial Estimates for the Method of Cyclic Projections in Hilbert Spaces

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$\mathcal{H}$ – real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$.

M_i – closed and linear subspaces of $\mathcal{H}$, $i = 1, \ldots, m$.

$M := \bigcap_{i=1}^m M_i$.

P_{M_i}, P_M – orthogonal projections onto M_i and M, respectively.

The Method of Cyclic Projections

$$y_0 \in \mathcal{H}, \qquad y_k := (P_{M_m} \ldots P_{M_1})^k (y_0), \qquad k = 1, 2, \ldots.$$

Rate of convergence for

$$\underbrace{\|y_k - P_M(y_0)\|, \qquad \sqrt{\frac{1}{m}\sum_{i=1}^m d^2(y_k, M_i)}, \qquad \|y_k - y_{k-1}\|}_{\to 0}.$$

[J. von Neumann (1933), I. Halperin (1962).]

Assumption

$$\sum_{i=1}^m M_i^\perp \text{ is not closed.}$$

Outline

Known Results

- What happens when $\sum_{i=1}^{m} M_i^{\perp}$ is closed?
- What happens when $\sum_{i=1}^{m} M_i^{\perp}$ is not closed?

New Results

- A view on the MCP from the product space $\bigoplus_{i=1}^{m} \mathcal{H}$.
- Best possible estimates when $\sum_{i=1}^{m} M_i^{\perp}$ is not closed.

What happens when $\sum_{i=1}^{m} M_i^{\perp}$ is closed?

$$\sum_{i=1}^{m} M_i^{\perp} = \overline{\sum_{i=1}^{m} M_i^{\perp}} = M^{\perp}$$

Linear Convergence

Theorem 1. $\sum_{i=1}^{m} M_i^{\perp}$ is closed $\iff \underbrace{\|P_{M_m} \dots P_{M_1} - P_M\|}_{q} < 1.$

[H.H. Bauschke, J.M. Borwein and A.S. Lewis (1997), F. Deutsch (1984).]

$$\|y_k - P_M(y_0)\| = O(q^k),$$

$$\sqrt{\tfrac{1}{m}\textstyle\sum_{i=1}^{m} d^2(y_k, M_i)} \overset{(*)}{=} O(q^k),$$

$$\|y_k - y_{k-1}\| \overset{(*)}{=} O(q^k).$$

$$\max\left\{\sqrt{\tfrac{1}{m}\textstyle\sum_{i=1}^{m} d^2(y_k, M_i)}, \quad \|y_k - y_{k-1}\|\right\} \leq 2\|y_{k-1} - P_M(y_0)\| \qquad (*)$$

What happens when $\sum_{i=1}^m M_i^\perp$ is not closed?

$$\sum_{i=1}^m M_i^\perp \neq \overline{\sum_{i=1}^m M_i^\perp} = M^\perp$$

Arbitrarily Slow Convergence

Theorem 2. Assume that $\sum_{i=1}^m M_i^\perp$ is not closed. Then, for each sequence $(a_k)_{k=0}^\infty \subset (0,\infty)$ with $a_k \to 0$, there is $y_0 \in \mathcal{H}$ such that

$$\|y_k - P_M(y_0)\| \geq a_k$$

for all $k = 1, 2, \ldots$.

[H.H. Bauschke, F. Deutsch and H. Hundal (2009), (2010).]

- There is no such $p > 0$ that

$$\|y_k - P_M(y_0)\| = o(k^{-p})$$

  holds for all $y_0 \in \mathcal{H}$.

- What about y_0's restricted to a subspace $X \subset \mathcal{H}$?

Super Polynomially Fast Convergence

Theorem 3. Assume that $\sum_{i=1}^m M_i^\perp$ is not closed. Then there is a dense linear subspace X of $\mathcal{H}$ such that for all $y_0 \in X$ and for all $p > 0$, we have

$$\|y_k - P_M(y_0)\| = o(k^{-p}).$$

[C. Badea and D. Seifert (2016).]

Consequently, for all $y_0 \in X$ and for all $p > 0$, we have

$$\sqrt{\frac{1}{m}\sum_{i=1}^m d^2(y_k, M_i)} \overset{(*)}{=} o(k^{-p})$$

and

$$\|y_k - y_{k-1}\| \overset{(*)}{=} o(k^{-p}).$$

$$\max\left\{\sqrt{\frac{1}{m}\sum_{i=1}^m d^2(y_k, M_i)}, \quad \|y_k - y_{k-1}\|\right\} \le 2\|y_{k-1} - P_M(y_0)\| \qquad (*)$$

Polynomial Convergence

Theorem 4. For each $y_0 \in \sum_{i=1}^{m} M_i^{\perp}$ (so that $P_M(y_0) = 0$), we have

$$\|y_k\| = \mathcal{O}(k^{-1/(4m\sqrt{m}-2)}).$$

Moreover, when $m = 2$, then

$$\|y_k\| = \mathcal{O}(k^{-1/2})$$

and $k^{-1/2}$ cannot be replaced by $k^{-1/2-\varepsilon}$ for any $\varepsilon > 0$.

[P. Borodin and E. Kopecká (2020).]

Open Problem

- Can we show that $\|y_k\| = O(k^{-1/2})$ holds for $m \geq 3$?
- Find the best power in $\|y_k\| = O(k^{-p})$ for $m \geq 3$?

Polynomial Convergence

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[P. Borodin and E. Kopecká (2020).]

Example of M_1 and M_2 leading to optimality

- For each $\varepsilon > 0$ define $y_0 \in M_1^\perp + M_2^\perp$ such that

$$\|y_k\| \geq \frac{C(y_0)}{k^{1/2+\varepsilon}} \qquad k = 1, 2, \ldots.$$

- $M_1^\perp + M_2^\perp$ is not closed.

Polynomial Convergence

Theorem 4. For each $y_0 \in \sum_{i=1}^m M_i^\perp$ (so that $P_M(y_0) = 0$), we have

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[P. Borodin and E. Kopecká (2020).]

The Lower Bound Conjecture

- If $\sum_{i=1}^m M_i^\perp$ is not closed, then for each $\varepsilon > 0$ there is $y_0 \in \sum_{i=1}^m M_i^\perp$ such that

$$\|y_k\| \geq \frac{C(y_0)}{k^{1/2+\varepsilon}} \qquad k = 1, 2, \ldots.$$

Asymptotic Regularity

Theorem 5. For each $y_0 \in \mathcal{H}$, we have

$$\|y_k - y_{k-1}\| = o(k^{-1}).$$

[C. Badea and D. Seifert (2016), M. Crouzeix (2008).]

- The product of projections $T := P_{M_m} \dots P_{M_1}$ satisfies

$$\|T^k - T^{k-1}\| = \mathcal{O}(k^{-1}).$$

- Can we replace k^{-1} by $k^{-1-\varepsilon}$ for any $\varepsilon > 0$?
- What is the rate if we restrict y_0's only to $\sum_{i=1}^m M_i^\perp$?

New Results

A view on the MCP from the product space

Product Space Setup

$$\boldsymbol{H} := \bigoplus_{i=1}^m \mathcal{H}, \quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^m \frac{1}{m} \langle x_i, y_i \rangle, \quad \|\boldsymbol{x}\| = \sqrt{\frac{1}{m} \sum_{i=1}^m \|x_i\|^2}$$

$$\boldsymbol{C} := M_1 \times \ldots \times M_m \qquad \text{and} \qquad \boldsymbol{D} := \{\underbrace{(x, \ldots, x)}_{m \text{ times}} : x \in \mathcal{H}\}$$

Theorem 6. For $\boldsymbol{x} = (x_1, \ldots, x_m) \in \boldsymbol{H}$, we have

$$P_{\boldsymbol{C}}(\boldsymbol{x}) = \big(P_{M_1}(x_1), \ldots, P_{M_m}(x_m)\big)$$

and

$$P_{\boldsymbol{D}}(\boldsymbol{x}) = (s, \ldots, s), \quad \text{where} \quad s := \frac{1}{m} \sum_{i=1}^m x_i.$$

[G. Pierra (1984).]

$$P_{\boldsymbol{C}^\perp}(\boldsymbol{x}) = \big(P_{M_1^\perp}(x_1), \ldots, P_{M_m^\perp}(x_m)\big), \quad \text{where} \quad \boldsymbol{C}^\perp = M_1^\perp \times \ldots \times M_m^\perp$$

$$P_{\boldsymbol{M}_i}(\boldsymbol{x}) = \big(P_{M_i}(x_1), \ldots, P_{M_i}(x_m)\big), \quad \text{where} \quad \boldsymbol{M}_i := M_i \times \ldots \times M_i$$

The MCP Seen from the Product Space

$$\boldsymbol{T} := P_{\boldsymbol{M}_m} \dots P_{\boldsymbol{M}_1} \in B(\boldsymbol{H})$$

- For $y_0 \in \mathcal{H}$ and $\boldsymbol{x} := (y_0, \dots, y_0)$, we get

$$\|y_k - y_{k-1}\| = \|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1}) P_{\boldsymbol{D}}(\boldsymbol{x})\|$$

$$\sqrt{\tfrac{1}{m}\textstyle\sum_{i=1}^m d^2(y_k, M_i)} = \|P_{\boldsymbol{C}^\perp} \boldsymbol{T}^k P_{\boldsymbol{D}}(\boldsymbol{x})\|$$

- For $y_0 = \frac{1}{m}\sum_{i=1}^m x_i$ with $x_i \in M_i^\perp$ and $\boldsymbol{x} := (x_1, \dots, x_m)$, we get

$$\|y_k\| = \|\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}(\boldsymbol{x})\|$$

$$\|y_k - y_{k-1}\| = \|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1}) P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}(\boldsymbol{x})\|$$

$$\sqrt{\tfrac{1}{m}\textstyle\sum_{i=1}^m d^2(y_k, M_i)} = \|P_{\boldsymbol{C}^\perp} \boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}(\boldsymbol{x})\|$$

Polynomial Estimates

Lemma 7. We have:

(i) $\|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1})P_{\boldsymbol{D}}\| = \mathcal{O}(k^{-1})$;
(ii) $\|P_{\boldsymbol{C}^\perp}\boldsymbol{T}^k P_{\boldsymbol{D}}\| = \mathcal{O}(k^{-1/2})$;
(iii) $\|\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \mathcal{O}(k^{-1/2})$;
(iv) $\|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1})P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \mathcal{O}(k^{-3/2})$;
(v) $\|P_{\boldsymbol{C}^\perp}\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \mathcal{O}(k^{-1})$.

Proof.

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(ii) Follows from

$$\|P_{\boldsymbol{C}^\perp}\boldsymbol{T}^k P_{\boldsymbol{D}}\| \leq \sqrt{\tfrac{m}{2}\|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1})P_{\boldsymbol{D}}\|} = O(k^{-1/2}).$$

(iii) Because $\boldsymbol{T}$ and $\boldsymbol{T}^*$ commute with $P_{\boldsymbol{D}}$, we have

$$\|\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \|(\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp})^*\| = \|P_{\boldsymbol{C}^\perp}(\boldsymbol{T}^*)^k P_{\boldsymbol{D}}\| \overset{(ii) \text{ for } \boldsymbol{T}^*}{=} O(k^{-1/2}).$$

...

■

Thresholds are Critical

Lemma 8. Assume that one of the following conditions holds for some $\varepsilon > 0$:

(i) $\|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1})P_{\boldsymbol{D}}\| = \mathcal{O}(k^{-1-\varepsilon})$;

(ii) $\|P_{\boldsymbol{C}^\perp}\boldsymbol{T}^k P_{\boldsymbol{D}}\| = \mathcal{O}(k^{-1/2-\varepsilon})$;

(iii) $\|\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \mathcal{O}(k^{-1/2-\varepsilon})$;

(iv) $\|(\boldsymbol{T}^k - \boldsymbol{T}^{k-1})P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \mathcal{O}(k^{-3/2-\varepsilon})$;

(v) $\|P_{\boldsymbol{C}^\perp}\boldsymbol{T}^k P_{\boldsymbol{D}} P_{\boldsymbol{C}^\perp}\| = \mathcal{O}(k^{-1-\varepsilon})$;

Then $\sum_{i=1}^m M_i^\perp$ is closed and all of the rates become $O(q^k)$ for some $q \in (0,1)$.

$$\underbrace{\text{Lemma 7} \quad + \quad \text{Lemma 8}}_{\text{in } \boldsymbol{H}} \quad \Longrightarrow \quad \text{Best Possible Estimates in } \mathcal{H}$$

Main Result

Theorem 9. Assume that $\sum_{i=1}^m M_i^\perp$ is not closed. Then, for each $y_0 \in \mathcal{H}$, we have

$$\|y_k - y_{k-1}\| = o(k^{-1}) \tag{1}$$

and

$$\sqrt{\frac{1}{m}\sum_{i=1}^m d^2(y_k, M_i)} = o(k^{-1/2}). \tag{2}$$

Moreover, for each $y_0 \in \sum_{i=1}^m M_i^\perp$ (so that $P_M(y_0) = 0$), we have

$$\|y_k\| = \mathcal{O}(k^{-1/2}), \tag{3}$$

$$\|y_k - y_{k-1}\| = \mathcal{O}(k^{-3/2}) \tag{4}$$

and

$$\sqrt{\frac{1}{m}\sum_{i=1}^m d^2(y_k, M_i)} = \mathcal{O}(k^{-1}). \tag{5}$$

Furthermore, all of the above-mentioned rates are best possible as the corresponding polynomials $k^{1/2}$, k^1 and $k^{3/2}$ cannot be replaced by $k^{1/2+\varepsilon}$, $k^{1+\varepsilon}$ and $k^{3/2+\varepsilon}$, respectively, for any $\varepsilon > 0$.

Lower Bound Property

Corollary 10. Assume that $\sum_{i=1}^{m} M_i^{\perp}$ is not closed. Then for each $\varepsilon > 0$ and for each $C > 0$ there is $y_0 \in \sum_{i=1}^{m} M_i^{\perp}$ and a countably infinite subset of indices $K \subset \mathbb{N}$ such that the lower bound

$$\|y_k\| \geq Ck^{-1/2-\varepsilon}$$

holds for all $k \in K$.

- Can we show that the lower bound property holds for all sufficiently large k?

References

Linear convergence

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Arbitrarily slow convergence

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Polynomial convergence

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Thank you for your attention!