Mathematical Perspective of Control Theory- A Glimpse

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Contents

- Controllability of a linear system in finite dimensional spaces.
- Link controllability issues of linear dynamical system with solvability analysis.
- Controllability of a nonlinear system with complete solvability analysis.
- Switch from finite dimensional control to infinite dimensional control.
- Existence and controllability results for fractional evolution equations via integral contractors

- Controllability is one of the qualitative property of a control system which plays a crucial role in the analysis and design of control systems.
- In controllability of a system, we show the existence of a control function which steers the solution of the system from its initial state to the desired final state, where the initial and final states may vary over the entire space.
- Controllability analysis can be made in many problems like:
  (i) Rocket launching problem: Satellite control and control of aircraft
  (ii) Biological System: Sugar level in blood
  (iii) Defence: Missiles and anti-missiles problems
  (iv) Economy: Regulating inflation rate
  (v) Ecology: Predator-Prey system

Autonomous Linear Control System

$$\begin{aligned}\frac{dx(t)}{dt} =& x'(t) = Ax(t) + Bu(t), \quad t_0 \leq t \leq T, \\ & x(t_0) = x_0.\end{aligned} \tag{1}$$

where

- A and B are real or complex matrices of $n \times n$ and $n \times m$ respectively.
- For each $t \in [t_0, T]$, $x(t)$ is the state element of the state space $\mathbb{R}^n$.
- $u(t)$ is the control element of the control space $\mathbb{R}^m$.

Solution

$$x(t) = \exp(A(t - t_0))x_0 + \int_{t_0}^{t} \exp(A(t - s))Bu(s)ds, \quad t_0 \leq t \leq T. \tag{2}$$

Exact Controllability

A control system (1) is said to be exactly controllable in the interval $I = [t_0, T]$ if for every initial state x_0 and desired final state x_f, there exists a control $u(t)$ such that the solution $x(t)$ of the system corresponding to this control u satisfies $x(T) = x_f$.

Approximate Controllability

A control system (1) is said to be approximately controllable in the interval $I = [t_0, T]$ if for every initial state x_0 and desired final state x_f and every given number $\epsilon > 0$, there exists a control u such that the solution $x(t)$ of the system corresponding to this control satisfies $|x(T) - x_f| < \epsilon$.

Kalman's condition for controllability

The autonomous linear control system is controllable if and only if

$$\text{rank}[B, AB, A^2B, \cdots, A^{n-1}B] = n.$$

The matrix $[B, AB, A^2B, \cdots A^{n-1}B]$ is called the controllability matrix.

$*$ Kalman, R. E.; Controllability of linear systems, Contrib. Differ. Eqn. 1, 190-213, 1963.

Time varying or Non autonomous Linear control system

$$\begin{aligned} \frac{dx(t)}{dt} &= x'(t) = A(t)x(t) + B(t)u(t), \quad t_0 \le t \le T, \\ x(t_0) &= x_0. \end{aligned} \tag{3}$$

Solution

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)ds, \quad t_0 \le t \le T. \tag{4}$$

where $\Phi(t,t_0)$ is the State Transition Matrix.

Controllability of finite dimensional linear control system

The system (3) is said to be controllable over $[t_0; T]$, if for every pair of vectors $x_0, x_F \in R^n$, there exists a control $u \in L^2[t_0, T, R^m]$ such that the solution of the system satisfies $x(T) = x_F$, that is

$$x(T) = \Phi(T, t_0)x_0 + \int_{t_0}^{T} \Phi(T, s)Bu(s)ds.$$

Equivalently, find u such that

$$y_T = x(T) - \Phi(T, t_0)x_0 = \int_{t_0}^{T} \Phi(T, s)Bu(s)ds. \tag{5}$$

Solvability Problem

Define $L : U = L^2[t_0, T, \mathbb{R}^m] \to \mathbb{R}^n$ by

$$Lu = \int_{t_0}^{T} \Phi(T,s)Bu(s)ds. \tag{6}$$

The controllability problem (5) reduces to the surjectivity of the operator $L : U \to \mathbb{R}^n$ That is, given $y_T \in \mathbb{R}^n$, find $u \in U$ such that

$$Lu = y_T. \tag{7}$$

Equivalent Formulation

The Adjoint $L^* : \mathbb{R}^n \to U$ is defined as follows

$$[L^* z](t) = B^{\mathbf{T}} \phi(T,t)^{\mathbf{T}} z. \tag{8}$$

Instead of solving (7), we solve

$$LL^* z = y_T. \tag{9}$$

For $z \in \mathbb{R}^n$, given $y_T \in \mathbb{R}^n$.
Which implies $u = L^* z$, then solves (7).

The operator $LL^* : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\begin{aligned} LL^* z &= \left[\int_{t_0}^{T} \Phi(T,s) B B^{\mathbf{T}} \Phi^{\mathbf{T}}(T,s) ds \right] z \\ &= W_{t_0}^T z. \end{aligned} \tag{10}$$

The matrix

$$W_{t_0}^T = \left[\int_{t_0}^{T} \Phi(T,s) B B^{\mathbf{T}} \Phi^{\mathbf{T}}(T,s) ds \right] \tag{11}$$

is called Grammian matrix.

Theorem

The linear time varying system is exactly controllable if and only if the $n \times n$ symmetric controllability matrix

$$W_{t_0}^T = \int_{t_0}^{T} \Phi(T,s)B(s)B^{\boldsymbol{T}}(s)\Phi^{\boldsymbol{T}}(T,s)ds,$$

is nonsingular. In this case the control

$$u(t) = -B^{\boldsymbol{T}}(t)\Phi^{\boldsymbol{T}}(t_0,t)(W_{t_0}^T)^{-1}[x_0 - \Phi(t_0,T)x_f], \tag{12}$$

where $x(T) = x_f$.

∗ Curtain, R. F., and Zwart, H.; An introduction to infinite-dimensional linear systems theory, vol. 21 of Texts in Applied Mathematics. Springer- Verlag, New York, 1995.

Semilinear Control System

$$\begin{aligned}\frac{dx(t)}{dt} &= x'(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t_0 \le t \le T,\\ x(t_0) &= x_0,\end{aligned} \tag{13}$$

where

- $x(t) \in$ Banach space $\mathbb{X}$.
- $u(t) \in$ Banach space $\mathbb{U}$.
- $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is closed, linear and densely defined operator.
- $B : L_2[t_0, T : \mathbb{U}] \to L_2[t_0, T : \mathbb{X}]$ is a bounded linear operator, where $L_2[0, T : \mathbb{X}]$ and $L_2[0, T : \mathbb{U}]$ are function spaces.
- $f : [0, T] \times \mathbb{X} \to \mathbb{X}$ is nonlinear function.

For any control $u \in U$, there exists a unique mild solution $x(t)$ of the system (13) (under suitable conditions on f), which is given by the nonlinear integral equation

Mild Solution

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)[Bu(s) + f(s, x(s))]ds, \quad t_0 \leq t \leq T.$$

The operator $S(t), t \geq 0$ associated with the mild solution is known as **strongly continuous semigroup** generated by the operator A.

Reachable Set of the Semilinear Control System

$$\mathcal{R}_T(f) \quad = \{x(T) \in \mathbb{X} : \text{x(t) is a mild solution of the semilinear control system corresponding to control u} \in L_2[t_0, T : \mathbb{U}]\}$$

$\mathcal{R}_T(0)$ is the reachable set of the corresponding linear control system.

Approximate Controllability

The semilinear system is said to be approximately controllable on $[t_0, T]$ if and only if $\mathcal{R}_T(f)$ is dense in $\mathbb{X}$, that means $\overline{\mathcal{R}_T(f)} = \mathbb{X}$. The corresponding linear system is approximately controllable if $\overline{\mathcal{R}_T(0)} = \mathbb{X}$.

Exact Controllability

The semilinear control system is said to be exactly controllable on $[t_0, T]$ if and only if $\mathcal{R}_T(f) = \mathbb{X}$. The corresponding linear system is exactly controllable if $\mathcal{R}_T(0) = \mathbb{X}$.

An important result on the controllability of semilinear systems by Naito, 1987

1. The semigroup $S(t)$ is compact,
2. The nonlinear function $f(t,x)$ is Lipschitz continuous,
3. $||f(t,x)|| \leq M$, where M is a positive constant,
4. For every $p \in Z = L_2[0,T;\mathbb{X}]$, there exists a $q \in \overline{R(B)}$ such that $Lp = Lq$, where the operator $L : Z \to \mathbb{X}$ is defined as

$$Lz = \int_0^T S(T-s)z(s)ds.$$

Condition (4) of the above theorem implies that the corresponding linear system is approximately controllable.

∗ Naito, K.; Controllability of semilinear control systems dominated by the linear part. SIAM J. Control and Optimization 1987, Vol. 25, 715-722.

Atangana-Baleanu Fractional Evolution Equation

$$\begin{aligned} &D^{\alpha}_{0+}x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in (0, T], \\ &x(0) = x_0, \end{aligned} \tag{14}$$

where

- D^{α}_{0+} represents the A-B fractional derivative of order $\alpha \in (0, 1)$ in the Caputo sense.
- The state variable $x(\cdot)$ belongs to the Banach space $(X, \|\cdot\|)$.
- The linear operator $A : \mathcal{D}(A) \subset X \to X$ is infinitesimal generator of a $\alpha-$ resolvent family $(\mathcal{T}_{\alpha}(t))_{t \geq 0}$.
- U be the Banach space of admissible control functions such that the control function $u(\cdot) \in L^2([0, T], U)$.
- $B : L^2([0, T], U) \to L^2([0, T], X)$ denotes a bounded linear operator.
- $f : (0, T) \times X \to X$ is a given nonlinear function.

Mittag-Leffler Function

The ML function of one parameter family given by

$$M_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad Re(\alpha) > 0,\ x \in \mathbb{C}(\text{the set of complex numbers}). \tag{15}$$

Atangana-Baleanu Fractional Derivative

The A-B fractional derivative of a function x of order $\alpha \in (0, 1)$ in Caputo sense is defined as

$$D^{\alpha}_{0+}x(t) = \frac{E(\alpha)}{1-\alpha}\int_0^t x'(s)M_\alpha(-\eta(t-s)^\alpha)ds, \quad 0 < t \leq T, \tag{16}$$

where $E(\alpha) = (1-\alpha) + \alpha/\Gamma(\alpha)$ denotes a normalization function that satisfies $E(0) = E(1) = 1$, M_α denotes the Mittag-Leffler function with $\eta = \frac{\alpha}{1-\alpha}$.

Resolvent Set

For a linear operator A, the set $\rho(A) := \{\mu \in \mathbb{C} : (\mu I - A) \text{ is invertible}\}$ is called resolvent set and the family $\mathcal{R}(\mu, A) := (\mu I - A)^{-1}$ is called the resolvent of A.

Sectorial Operator

A closed linear operator A is called sectorial operator if for $\sigma \in \mathbb{R}$ and $\theta \in [\frac{\pi}{2}, \pi]$, there exist $\lambda > 0$ such that

(i) $\rho(A) \subset \sum_{\theta,\sigma} = \{\mu \in \mathbb{C} : \mu \neq \sigma, |\arg(\mu - \sigma)| < \theta\}$,

(ii) $\|\mathcal{R}(\mu, A)\| \leq \frac{\lambda}{|\mu - \sigma|}$, $\mu \in \sum_{\theta,\sigma}$.

$\ast$ Pazy, A.; Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44, Springer, New York, 1983.

Mild Solution

For given sectorial operator A, the mild solution $x(t)$ of Equation (14) is defined as

$$x(t) = R\mathcal{S}_\alpha(t)x_0 + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Bw(s)+f(s,x(s))]ds$$
$$+ \frac{\alpha R^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)[Bw(s)+f(s,x(s))]ds, \tag{17}$$

$\forall\, t \in [0,T]$, where R and V are linear operators given by

$$R = \sigma(\sigma I - A)^{-1}, \;\; V = \eta A(\sigma I - A)^{-1} \;\text{ with }\; \sigma = \frac{E(\alpha)}{1-\alpha},$$
$$\mathcal{S}_\alpha(t) = M_\alpha(-Vt^\alpha) = \frac{1}{2\pi i}\int_c e^{st}s^{\alpha-1}(s^\alpha I - V)^{-1}ds,$$

and

$$\mathcal{T}_\alpha(t) = t^{\alpha-1}M_{\alpha,\alpha}(-Vt^\alpha) = \frac{1}{2\pi i}\int_c e^{st}(s^\alpha I - V)^{-1}ds,$$

where c denotes a certain path lying on $\sum_{\theta,\sigma}$.

Reachable Set

Consider the reachable set $K_T(f, u) := \{x(T, x_0, u) : u(\cdot) \in L^2([0, T], U)\}$ of (14) which is collection of all final states x at terminal time T with initial state x_0 and control u.

Exact Controllability

The fractional evolution equation (14) is said to be exactly controllable on $[0, T]$ if and only if $K_T(f, u) = \mathcal{D}(A)$.

Integral Contractor

A bounded linear operator $\Phi : [0, T] \times X \to \mathcal{B}(X)$ is called a bounded integral contractor of function f with respect to the operator $\mathcal{T}_\alpha(t)$ if there exists a constant $\tau > 0$ such that

$$\begin{aligned}\Bigg\| f\Bigg(t, x(t) + y(t) + &\frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Phi(s, x(s))y(s)ds \\ &+ \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s)\Phi(s, x(s))y(s)ds \Bigg) \\ &- f(t, x(t)) - \Phi(t, x(t))y(t) \Bigg\| \le \tau \|y(t)\|, \end{aligned} \tag{18}$$

for all $t \in (0, T)$ and $x, y \in X$.

$*$ George R. K.; Approximate controllability of semilinear systems using integral contractors, *Numer. Funct. Anal. Optim.*, 16(1995), 127-138.

Regular Integral Contractor

if for any $x, z \in X$, the integral equation

$$z(t) = y(t) + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Phi(s, x(s)) y(s) ds + \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s) \Phi(s, x(s)) y(s) ds \tag{19}$$

admits a solution $y \in X$, then Φ is called a regular integral contractor.

∗ George R. K.; Approximate controllability of semilinear systems using integral contractors, *Numer. Funct. Anal. Optim.*, 16(1995), 127-138.

Remark

For the case $\Phi \equiv 0$,the nonlinear function $f(t,x(t))$ has to satisfy the following Lipschitz-type condition:

$$\|f(t,x(t)+y(t)) - f(t,x(t))\| \leq \tau\|y(t)\|, \tag{20}$$

In other words, if f satisfies this condition, then it has the regular integral contractor $\Phi \equiv 0$. Thus, the results obtained in the present paper are also valid for those functions which satisfy this Lipschitz-type condition.

Generalized Gronwall's inequality

Let $x(t)$ and $c(t)$, $t \in [0, T)$, be two nonnegative locally integrable functions such that

$$x(t) \leq c(t) + h \int_0^t (t-s)^{\alpha-1} x(s) ds \qquad (21)$$

for some $h \geq 0$ and $\alpha > 0$. Then

$$x(t) \leq c(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(h\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} c(s) \right] ds, \quad 0 \leq t < T. \qquad (22)$$

$\ast$ Haiping Y., Jianming G. and Yongsheng D.; A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* 328 (2007), No. 2, 1075-1081.

To find the existence of solution, we consider the following assumptions:

(H_1) A is a sectorial operator.

(H_2) R and V are bounded linear operators such that $\|R\| \leq k_1$ and $\|V\| \leq k_2$, where k_1 and k_2 are positive constants.

(H_3) The nonlinear function $f : [0, T] \times X \to X$ fulfills the subsequent conditions

- (i) f has a regular integral contractor Φ.
- (ii) $f(\cdot, x) : [0, T] \to X$ is measurable for every $x \in X$;
- (iii) $f(t, \cdot) : X \to X$ is continuous for almost every $t \in [0, T]$;

Theorem

If $(H_1) - (H_3)$ holds true, the fractional evolution equation (14) has a unique mild solution.

Outline of the Proof

Consider the two sequences $\{x_n\}$ and $\{y_n\}$ in X defined as

$$x_0(t) = R\mathcal{S}_\alpha(t)x_0 + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Bw(s)ds + \frac{\alpha R^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)Bw(s)ds, \quad (23)$$

$$\begin{aligned} y_n(t) =& x_n(t) - \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x_n(s))ds \\ &- \frac{\alpha R^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)f(s,x_n(s))ds - x_0(t), \end{aligned} \quad (24)$$

$$\begin{aligned} x_{n+1}(t) =& x_n(t) - \Big[y_n(t) + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Phi(s,x_n(s))y_n(s)ds \\ &+ \frac{\alpha R^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)\Phi(s,x_n(s))y_n(s)ds\Big]. \end{aligned} \quad (25)$$

Step 1: The sequence $\{y_n\}$ converges to zero as $n \to \infty$.
Using Equation (24) and (25)

$$x_{n+1}(t) = \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Phi(s, x_n(s))y_n(s)ds + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s))ds \\ - \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s)\Phi(s, x_n(s))y_n(s)ds + \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s) f(s, x_n(s))ds + x_0(t).$$

Again, from Equation (24)

$$\|y_{n+1}(t)\| \leq \left[\frac{k_1 k_2 \tau(1-\alpha)}{E(\alpha)\Gamma(\alpha)} + \frac{\alpha\tau(k_1)^2 D_{\mathcal{T}}}{E(\alpha)}\right] \int_0^t (t-s)^{\alpha-1}\|y_n(s)\|ds.$$

By induction, we obtain

$$\|y_{n+1}\| \leq \frac{\left[\frac{T(k_1 k_2 \tau(1-\alpha) + \Gamma(1+\alpha)\tau(k_1)^2 D_{\mathcal{T}})}{E(\alpha)}\right]^{n+1}}{\Gamma(1+(n+1)\alpha)} \|y_0\|. \tag{26}$$

Let

$$p = \frac{T(k_1 k_2 \tau(1-\alpha) + \Gamma(1+\alpha)\tau(k_1)^2 D_{\mathcal{T}})}{E(\alpha)}.$$

Then

$$M_\alpha(p) = \sum_{n=0}^{\infty} \frac{p^n}{\Gamma(1+n\alpha)}.$$

From the convergence of Mittag-Leffler function of order α at point p, we obtain that $\{y_n\} \to 0$ as $n \to \infty$ in X.

Step 2: $\{x_n\}$ is a Cauchy sequence in X which converges to a point in X.
From Equation (25) and (26)

$$\|x_{n+1}(t) - x_n(t)\| \leq \left[1 + \frac{k_1 k_2 \nu(1-\alpha)T^\alpha}{E(\alpha)\Gamma(1+\alpha)} + \frac{T^\alpha (k_1)^2 D_{\mathcal{T}}}{E(\alpha)}\right] \frac{p^n}{\Gamma(1+n\alpha)} \|q_0\|.$$

As such, we have

$$\begin{aligned}\|x_n - x_m\| &\leq \sum_{k=m}^{n-1} \|x_{k+1} - x_k\| \\ &\leq \left[1 + \frac{k_1 k_2 \nu(1-\alpha)T^\alpha}{E(\alpha)\Gamma(1+\alpha)} + \frac{T^\alpha (k_1)^2 D_{\mathcal{T}}}{E(\alpha)}\right] \|q_0\| \sum_{k=m}^{n-1} \frac{p^n}{\Gamma(1+n\alpha)},\end{aligned}$$

for $n > m \geq 0$. Hence $\{x_n\}$ is a Cauchy sequence in X which will converge to a point say x^* in X.

Using equation (24) and the well known Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} x_n(t) - \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \lim_{n\to\infty} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s))ds$$
$$- \frac{\alpha R^2}{E(\alpha)} \lim_{n\to\infty} \int_0^t \mathcal{T}_\alpha(t-s) f(s, x_n(s))ds - x_0(t),$$

which implies

$$x^*(t) = R\mathcal{S}_\alpha(t)x_0 + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Bw(s) + f(s, x^*(s))]ds$$
$$+ \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s)[Bw(s) + f(s, x^*(s))]ds.$$

Which proves that x^* is the mild solution of Equation (14).

Step 3: To prove the uniqueness of solution by utilizing the regularity property of the integral contractor. For fixed control $u \in L^2([0,T],U)$, let x_1 and x_2 be two solutions of Equation (14). Then

$$x_2(t) - x_1(t) = \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[f(s,x_2(s)) - f(s,x_1(s))]ds \quad (27)$$

$$+ \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s)[f(s,x_2(s)) - f(s,x_1(s))]ds. \quad (28)$$

Using definition of regular integral contractor and equation (19), we obtain

$$\|x_2(t) - x_1(t)\| \leq \left[\frac{k_1 k_2 \tau(1-\alpha)}{E(\alpha)\Gamma(\alpha)} + \frac{\alpha\tau D_{\mathcal{T}}(k_1)^2}{E(\alpha)}\right] \int_0^t (t-s)^{\alpha-1}\|x_2(s) - x_1(s)\|ds.$$

Using Generalized Gronwall inequality for fractional fractional differential equations, we get $\|x_2(t) - x_1(t)\| = 0$ for any $t \in [0,T]$ i.e $x_1 = x_2$. Hence the solution of Equation (14) is unique.

Assumptions

(H_4) The linear equation corresponding to Equation (14)

$$\begin{aligned} D^{\alpha}_{0+}x(t) =& Ax(t) + Bw(t), \quad t \in (0,T], \\ x(0) =& x_0, \end{aligned} \tag{29}$$

is exact controllable with control w.

(H_5) $\mathcal{R}(f) \subseteq \mathcal{R}(B)$.

Theorem

If assumptions $(H_1) - (H_5)$ hold true, then the fractional evolution equation (14) is exactly controllable.

Outline of the proof

Consider the linear A-B fractional evolution equation

$$\begin{aligned} &D^{\alpha}_{0+}y(t) = Ay(t) + Bw(t), \quad t \in (0,T], \\ &y(0) = y_0 = \varphi. \end{aligned} \tag{30}$$

Using Definition (17), we obtain

$$y(t) = R\mathcal{S}_\alpha(t)\varphi + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Bw(s)ds + \frac{\alpha^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)Bw(s)ds. \tag{31}$$

Also, consider the perturbed equation

$$\begin{aligned} D^{\alpha}_{0+}x(t) =& Ax(t) + Bw(t) + f(t, x(t)) \\ & - f\left(t, y(t) + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Phi(s, x(s))(y-x)(s)ds \right. \\ & \left. + \frac{\alpha R^2}{E(\alpha)} \int_0^t \mathcal{T}_\alpha(t-s)\Phi(s, x(s))(y-x)(s)ds \right), \quad t \in (0, T], \qquad (32) \\ x(0) =& x_0 = \varphi, \end{aligned}$$

with mild solution

$$\begin{aligned} x(t) =& R\mathcal{S}_\alpha(t)\varphi + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[Bw(s) + f(s,x(s)) \\ & - f\Big(s, y(s) + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^s (s-\xi)^{\alpha-1}\Phi(\xi,x(\xi))(y-x)(\xi)d\xi \\ & + \frac{\alpha R^2}{E(\alpha)}\int_0^s \mathcal{T}_\alpha(s-\xi)\Phi(\xi,x(\xi))(y-x)(\xi)d\xi\Big)\Big]ds \\ & + \frac{\alpha R^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)\Big[Bw(s) + f(s,x(s)) \\ & - f\Big(s, y(s) + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^s (s-\xi)^{\alpha-1}\Phi(\xi,x(\xi))(y-x)(\xi)d\xi \\ & + \frac{\alpha R^2}{E(\alpha)}\int_0^s \mathcal{T}_\alpha(s-\xi)\Phi(\xi,x(\xi))(y-x)(\xi)d\xi\Big)\Big]ds, \end{aligned} \tag{33}$$

Equation (14) and (32) implies

$$\begin{aligned} Bu(t) =& Bw(t) - f\Bigg(t, y(t) + \frac{RV(1-\alpha)}{E(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Phi(s,x(s))(y-x)(s)ds \\ &+ \frac{\alpha R^2}{E(\alpha)}\int_0^t \mathcal{T}_\alpha(t-s)\Phi(s,x(s))(y-x)(s)ds\Bigg), \end{aligned} \tag{34}$$

which holds due to the assumption (H_5).

Also, subtracting Equation (33) from (31),

$$\|y(t)-x(t)\| \leq (\tau+\nu)\left[\frac{k_1k_2(1-\alpha)}{E(\alpha)\Gamma(\alpha)}+\frac{\alpha D_{\mathcal{T}}(k_1)^2}{E(\alpha)}\right]\int_0^t (t-s)^{\alpha-1}\|(y-x)(s)\|ds.$$

Using Generalized Gronwall inequality for fractional fractional differential equations, we obtain $\|y(t)-x(t)\|=0$ i.e. $\|y-x\|=0$. Hence $y(t)=x(t)$ for all $t\in[0,T]$.
Hence every mild solution of the linear Equation (30) is also a mild solution of semilinear Equation (32) which implies that $K_T(0,w)\subset K_T(f,u)$. Moreover, from assumption (H_4), we have $K_T(0,w)=D(A)$. Hence $K_T(f,u)=D(A)$, which assures that the fractional evolution equation (14) is exactly controllable over $[0,T]$.

Thank You