Hidden Positivity and a New Approach to Numerical Computation of Hausdorff Dimension

Roger Nussbaum
Dept. of Mathematics, Rutgers University
nussbaum@math.Rutgers.edu

Joint work with
Richard S. Falk
Dept. of Mathematics, Rutgers University
falk@math.Rutgers.edu

Continued Fractions; Invariant Sets

For $x \in (0,1]$, define $\left[\frac{1}{x}\right] := n :=$ greatest integer n with $n \le \left(\frac{1}{x}\right)$

Given $x \in (0,1]$ write $\frac{1}{x} = n_1 + x_1$, where $n_1 = \left[\frac{1}{x}\right]$ so

$$x = \frac{1}{n_1 + x_1}, \text{ where } 0 \le x_1 < 1$$

If $x_1 \ne 0$, we can write

$$x_1 = \frac{1}{n_2 + x_2}, \text{ where } n_2 = \left[\frac{1}{x_1}\right] \text{ and } 0 \le x_2 < 1; \text{ so}$$

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + x_2}}$$

If $x_2 \ne 0$, we can continue, etc.

The positive integers $n_1, n_2, \dots, n_k, \dots$ obtained in this way are the "terms" or "coefficients" of the continued fraction expansion of $x \in (0,1]$.

The continued fraction expansion of $x \in (0,1]$ has only finitely many terms iff x is rational.

The set $E[\mathcal{B}]$, for $\mathcal{B} \subset \mathbb{N}$

If $\mathcal{B}$ is a finite set of positive integers, define

$$E[\mathcal{B}] := \{x \in [0,1] \setminus \mathbb{Q} \mid \text{all terms in the contined fraction expansion of } x \text{ lie in } \mathcal{B}\}$$

<u>PROBLEM</u> Find high order <u>RIGOROUS</u> approximations for $\dim_H(E[\mathcal{B}])$

A Generalization of $E[\mathcal{B}]$

$\forall\, b \in (0,\infty)$, $\theta_b : [0,\infty) \to [0,\infty)$ is defined by

$$\boxed{\theta_b(x) = \frac{1}{b+x}}$$

$\mathcal{B}$:= finite set of positive reals

$$\boxed{\gamma := \min\{b \mid b \in \mathcal{B}\}, \quad \Gamma := \max\{b \mid b \in \mathcal{B}\}}$$

<u>Theorem</u> $\exists$ a <u>unique</u>, nonempty compact set $K[\mathcal{B}]$ such that

$$\boxed{K[\mathcal{B}] = \bigcup_{b \in \mathcal{B}} \theta_b(K[\mathcal{B}])}$$

Also, if $\mathcal{B} \subset \mathbb{N}$,

$$\boxed{K[\mathcal{B}] = E[\mathcal{B}]}$$

Locating $K[\mathcal{B}]$

$\mathcal{B}$ is a finite set $\subset (0,\infty)$

$\gamma = \min\{b \mid b \in \mathcal{B}\}$, $\Gamma = \max\{b \mid b \in \mathcal{B}\}$.

$\mathcal{B}_m = \{ \underbrace{(b_1, b_2, \dots, b_m)}_{\omega} \mid b_j \in \mathcal{B},\ 1 \le j \le m \}$

Given $\omega \in \mathcal{B}_m$, $\omega = (b_1, b_2, \dots, b_m)$, $\theta_\omega = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_m}$

where $\theta_b(x) := \frac{1}{x+b}$

← Constant Notation

Define

$$\alpha = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 + \left(\frac{\gamma}{\Gamma}\right)}, \quad \beta = -\frac{\Gamma}{2} + \sqrt{\left(\frac{\Gamma}{2}\right)^2 + \left(\frac{\Gamma}{\gamma}\right)} = \left(\frac{\Gamma}{\gamma}\right)\alpha$$

and $J = [\alpha, \beta]$.

Then

$\theta_b(J) \subset J \ \forall b \in \mathcal{B}$ and $K[\mathcal{B}] \subset J$

$E[1,2]$	14	0.0002	7	
	0.531 280 506 277 205 141 624 468 647 368 471 785 493 059 109 018 398			
$E[1,3]$	8	5.0e-05	6	
	0.454 489 077 661 828 743 845 777 611 651			
$E[1,4]$	8	5.0e-05	6	
	0.411 182 724 774 791 776 844 805 904 696			
$E[1,2,3]$	5	0.0001	5	
	0.705 660 908 028 738			
$E[1,3,5]$	8	0.001	6	
	0.581 366 821 182 975			
$E[2,3,4,5]$	16	0.005	4	
	0.559 636 450 164 776 713 312 144 913 530			
$E[1,2,3,4,5]$	5	0.0005	5	
	0.836 829 443 681 209			
$E[1,3,5,\ldots,33]$	10	0.01	1*	
	0.770 516 008 717 163			

$\theta_b(K[\mathcal{B}]) \cap \theta_c(K[\mathcal{B}]) = \phi$ for $b, c\,\mathcal{B}$, $b \neq c$??

$\mathcal{B} \subset (0, \infty)$ is a finite set; $\gamma, \Gamma \in \mathcal{B}$ and

$\gamma \leq b \leq \Gamma \quad \forall\, b \in \mathcal{B}$

A sufficient condition that

$\theta_b(K[\mathcal{B}]) \cap \theta_c(K[\mathcal{B}]) = \phi \quad \forall\, b, c \in \mathcal{B}$, $b \neq c$ is

$(*) \quad |c-b| > \left(\frac{1}{\gamma} - \frac{1}{\Gamma}\right)\left[\frac{2}{1+\sqrt{1+\frac{4}{\gamma\Gamma}}}\right] \quad \forall\, b, c \in \mathcal{B},\ b \neq c$

$(*)$ is satisfied if

$(**) \quad c - b \geq \frac{1}{\gamma}$ whenever $c, b \in \mathcal{B}$ $c > b$

$(**)$ is satisfied if $\mathcal{B} \subset \mathbb{N}$

$\mathrm{Dim}_H(K[B])$ and the Operator L_s, $s \geq 0$

Let S be a compact interval, $S \subset [0, \infty)$.

Assume that $\theta_b(S) \subset S \ \forall b \in B$.

For $s \geq 0$, define $L_s : X := C_{\mathbb{R}}(S) \to X$ by

$$\boxed{(L_s f)(x) = \sum_{b \in B} |\theta_b'(x)|^s f(\theta_b(x))}$$

The map $s \to \lambda_s := R(L_s) =$ spectral radius of L_s

is C^∞, strictly decreasing and log convex.

If $K := K[B]$ and $\theta_b(K) \cap \theta_c(K) = \phi \ \forall b, c \in B$, $b \neq c$,

$$\boxed{\dim_H(K[B]) = s_* \text{, where } \lambda_{s_*} = 1}$$

The Spectrum of Λ_s

Take $S \subset [0,\infty)$ to be a compact interval, $\theta_b(S) \subset S$ $\forall b \in \mathcal{B}$

Let Y denote one of $\mathrm{Lip}_{\mathbb{R}}(S)$ or $C^n_{\mathbb{R}}(S)$ for some $n \in \mathbb{N}$

Then $L_s|Y$ defines a bounded linear operator $\Lambda_s : Y \to Y$.

(i) $R(\Lambda_s) = R(L_s) := \lambda_s > 0$, where $L_s : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$

(ii) $\varrho(\Lambda_s) := \sup\left\{\frac{|z|}{\lambda_s} : z \in \sigma(\Lambda_s) \setminus \{\lambda_s\}\right\} < 1$

(iii) λ_s is an algebraically simple eigenvalue of Λ_s with eigenvector w_s; $w_s(x) > 0$ $\forall x \in S$ and $w_s \in Y$ is unique (to within positive scalar multiples).

Convergence of iterates of $\left(\frac{1}{\lambda_s}\Lambda_s\right)$

If $u \in Y$ ($Y = \mathrm{Lip}_{\mathbb{R}}(S)$ or $Y = C^n_{\mathbb{R}}(S)$) and $u(x) > 0$ $\forall x \in S$, $\exists\ a = a(u) > 0$ such that

$$(\star) \qquad \lim_{n\to\infty} \left(\frac{1}{\lambda_s}\Lambda_s\right)^n (u) = a w_s .$$

Convergence in $(\star)$ is in the Y topology.
Here w_s is the unique, normalized strictly positive eigenvector of Λ_s and $\lambda_s := R(\Lambda_s)$

Estimates for $\frac{D^k v_s(x)}{v_s(x)}$, $x \in [\alpha, \beta]$, $D = \frac{d}{dx}$

There is a unique strictly positive eigenvector (to within scalar multiples) $v_s \in C^n_{\mathbb{R}}([\alpha, \beta])$ for $\Lambda_s : Y := C^n_{\mathbb{R}}([\alpha,\beta]) \to Y$, $\Lambda_s(f) = L_s(f)$ for $f \in Y$.

The eigenvalue for v_s is $R(L_s) = R(\Lambda_s)$

(Here $\alpha := -\frac{\gamma}{2} + \sqrt{(\frac{\gamma}{2})^2 + (\frac{\gamma}{\Gamma})}$, $\beta := (\frac{\Gamma}{\gamma})\alpha$)

We have the following estimates for $x \in [\alpha, \beta]$:

$$(-1)^k \left[\frac{D^k v_s(x)}{v_s(x)} \right] \leq \frac{(2s)(2s+1)\cdots(2s+k-1)}{\left(\gamma \sqrt{1 + (4/\gamma\Gamma)}\right)^k}$$

$$\frac{(2s)(2s+1)\cdots(2s+k-1)}{\left(\Gamma \sqrt{1 + (4/\gamma\Gamma)}\right)^k} \leq (-1)^k \left[\frac{D^k v_s(x)}{v_s(x)} \right]$$

Iterating Λ_s

For $f \in Y$,

$$(\Lambda_s f)(x) = \sum_{b \in \mathcal{B}} |\theta_b'(x)|^s f(\theta_b(x))$$

One can check that

$$(\Lambda_s^m f)(x) = \sum_{\omega \in \mathcal{B}_m} |\theta_\omega'(x)|^s f(\theta_\omega(x))$$

(Recall that $\omega = (b_1, b_2, \dots, b_m)$ and $\theta_\omega = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_m}$)

<u>Remark</u>. $(R(\Lambda_s))^m = R(\Lambda_s^m)$, so

$$R(\Lambda_s^m) = 1 \iff R(\Lambda_s) = 1.$$

Cones and Positive Linear Operators

Let X be a real Banach space.

If $K \subset X$, K is called a "cone" if

(i) $\alpha x + \beta y \in K$ whenever $x, y \in K$ and $\alpha, \beta \in \mathbb{R}$ are nonegative

and

(ii) If $x \in C \setminus \{0\}$, then $-x \notin C$

If K is closed in the topology on X, K is a "closed cone".

An Important Example: The Cone $K_M(T)$

Let T = a bounded complete metric space with metric ρ

Assume T contains at least 2 points

$Y = \mathrm{Lip}_{\mathbb{R}}(T) :=$ B-space of Lipschitz maps $f : T \to \mathbb{R}$

$X =$ B-space of bounded continuous maps $f : T \to \mathbb{R}$

$K_M(T) := \{ f \in X \mid f(t_1) \leq f(t_2) \exp(M\rho(t_1, t_2)) \ \forall t_1, t_2 \in T \} \subset Y$

$K_M(T)$ is a closed "normal cone" in Y and

$\mathrm{int}(K_M(T))$ in Y is nonempty

$\sigma(A)$ if $A: Y \to Y$ and $A(K_M(T)) \subset K_{M'}(T)$, $M' < M$?

(T, ρ) is a bounded, complete metric space; $Y := Lip_{\mathbb{R}}(T)$

$A: Y \to Y$ is a bounded linear map; $A^2 \neq 0$.

<u>Theorem</u> Assume also that $A(K_M(T)) \subset K_{M'}(T)$, $M' < M$.
Then

(1) $\exists\, v \in K_{M'}(T) \setminus \{0\}$ with $Av = R(A)v$. $R(A) > 0$ and $R(A)$ has algebraic multiplicity equal to 1.

(2) $\sigma(A) \setminus \{R(A)\} \subseteq \{z \in \mathbb{C} \mid |z| \le R'\}$, where $R' < R(A)$.

(3) $u \in K_M(T)$ and $A(u) \neq 0 \Rightarrow \exists\, a = a(u) > 0$ with

$$\lim_{j \to \infty} \left(\frac{1}{R(A)} A\right)^j (u) = a(u)\, v$$

Piecewise Polynomials; the Collocation Method

Take $N \in \mathbb{N}$ large, $h := \frac{\beta - \alpha}{N}$ and $t_i = \alpha + ih$, $0 \le i \le N$

Select r = positive integer and $c_{i,j}$, $1 \le i \le N$, $0 \le j \le r$, with

$$c_{i,0} = t_{i-1} < c_{i,1} < c_{i,2} < \cdots < c_{i,r} = t_i$$

Define $T = \{ c_{ij} \mid 1 \le i \le N, 0 \le j \le r \}$; $|T| = Nr + 1$ and

$C_{\mathbb{R}}(T) = \{ f : T \to \mathbb{R} \}$ = $Nr+1$ dimensional vector space

If $f : T \to \mathbb{R}$, there is a unique polynomial

$p_i(t)$, $t \in [t_{i-1}, t_i]$, such that degree$(p_i) \le r$ and

$$p_i(c_{i,j}) = f(c_{i,j}), \quad 0 \le j \le r$$

Define $\mathcal{F} : [\alpha, \beta] \to \mathbb{R}$

$$\mathcal{F}(t) = p_i(t) \text{ for } t \in [t_{i-1}, t_i], \quad 1 \le i \le N$$

Extended Chebyshev Points on $[-1,1]$

Define $\hat{c}_k = -\left[\cos\left(\frac{2k+1}{2r+2}\pi\right) \Big/ \cos\left(\frac{\pi}{2r+2}\right) \right]$

Important to define

$$c_{j,k} = t_{j-1} + \left(\frac{h}{2}\right)(1+\hat{c}_k) \; ; \; 1 \le j \le N, \; 0 \le k \le r$$

If $x \in [t_{j-1}, t_j]$, $x = t_{j-1} + \frac{h}{2}(1+\hat{x})$, $-1 \le \hat{x} \le 1$

If $f \in C_{\mathbb{R}}(T)$ and f extends to a C^{r+1} function on each interval $[t_{j-1}, t_j]$, $1 \le j \le N$, then

$$|f(x) - \mathcal{F}(x)| \le \left[\frac{1}{(r+1)!}\right] \left[\max_{x \in [\alpha,\beta]} |f^{(r+1)}(x)|\right] \left(\frac{h}{2}\right)^{r+1} \left| \prod_{k=0}^{r} (\hat{x} - \hat{c}_k) \right|$$

$$\forall x \in [\alpha, \beta]$$

$$\max_{-1 \le \hat{x} \le 1} \left| \prod_{k=0}^{r} (\hat{x} - \hat{c}_k) \right| = \left(\frac{1}{2^r}\right) \left[\frac{1}{\cos\left(\frac{\pi}{2r+2}\right)}\right]^{r+1}$$

Defining the "Approximation" $A_{s,n}$ to L_s^n

For $f \in C_{\mathbb{R}}([\alpha,\beta])$

$$(L_s^n f)(x) = \sum_{\omega \in B_n} |\theta_\omega'(x)|^s f(\theta_\omega(x))$$

For $T = \{ c_{i,j} \mid 1 \le i \le N, 0 \le j \le r \}$ and $f \in C_{\mathbb{R}}(T)$

we have defined $\mathcal{F} : [\alpha,\beta] \to \mathbb{R}$ by $\mathcal{F}|[t_{i-1}, t_i] = p_i$,

where $p_i(c_{ij}) = f(c_{ij})$, $0 \le j \le r$ and p_i is a polynomial

of degree $\le r$. We define $g \in C_{\mathbb{R}}(T)$ by

$$g(c_{ij}) = \sum_{\omega \in B_n} |\theta_\omega'(c_{ij})|^s \mathcal{F}(\theta_\omega(c_{ij}))$$

The map $f \in C_{\mathbb{R}}(T) \to g \in C_{\mathbb{R}}(T)$ defines a linear

map $A_{s,n} : C_{\mathbb{R}}(T) \to C_{\mathbb{R}}(T)$

Approximating $R(L_s^n)$ by $R(A_{s,n})$

Assume that $r \geq 2$ is a given positive integer and $0 \leq s \leq 3/2$. There exists a positive integer n and $h_0 > 0$ such that if $h = \frac{\beta - \alpha}{N} \leq h_0$ then $\exists$ constants $M(r, s, h_0) > 0$ and $M' = M'(r, s, h_0) < M$, $M' > 0$, with

$$A_{s,n}(K_M(T)) \subset K_{M'}(T), \quad A_{s,n}^2 \neq 0$$

There is a constant $H = H(r, s, h_0) > 0$ such that for $\lambda_s^n = R(L_s^n)$

$$(*) \quad \lambda_s^n (1 - H h^r) \leq R(A_{s,n}) \leq \lambda_s^n (1 + H h^r)$$