

A primal/dual computable approach to algorithmic improvement, based on dynamics

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A Workshop on Nonlinear Functional Analysis and Its Applications
In memory of Professor Ronald E. Bruck
4-6 April 2022

Last Revised April 4, 2022

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Outline

Splitting Methods

Centering

Duality

Centering and Lyapunov Functions

Summary

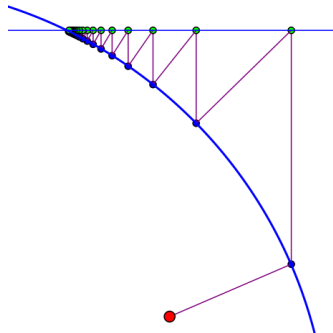
00: Splitting Algorithms

Problem:

$$\underset{x \in E}{\text{minimize}} \quad f(x) + g(z) \quad \text{s.t.} \quad Mx = z.$$

Splitting Methods break a problem into more easily computable subproblems by:

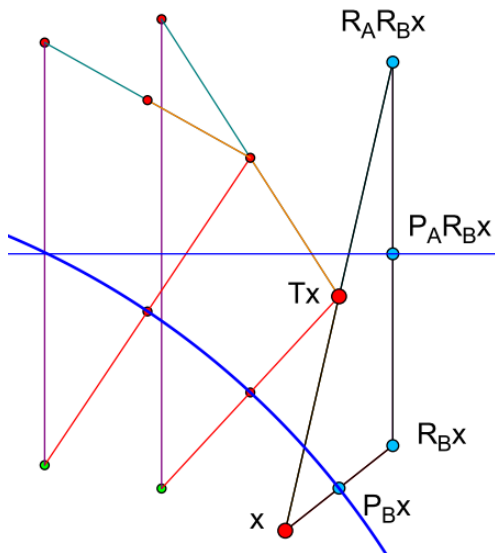
1. **decoupling constraints** that are difficult to simultaneously satisfy
2. **decoupling minimization steps** that are difficult to simultaneously compute
3. **decoupling** a **constraint** satisfaction step from a **minimization** step



01: Douglas–Rachford

Popular variants include:

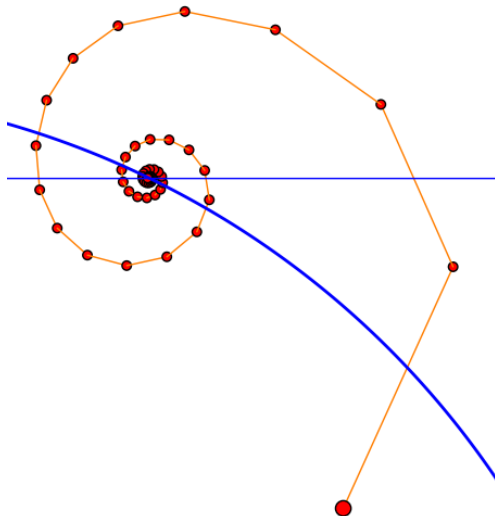
1. ADMM and its dual
Douglas–Rachford
(right). Recent survey
of DR by Lindstrom
and Sims (2018)
2. Proximal gradient
(forward–backward
method), (e.g.
alternating
projections)
3. FISTA



02: Spiraling Algorithms

Algorithms that may spiral include:

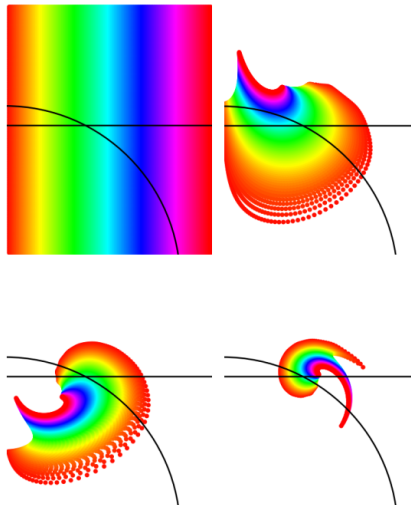
1. Douglas–Rachford
(dual to ADMM)
2. FISTA



03: Dynamical systems

Case study: sphere and line
Douglas–Rachford:

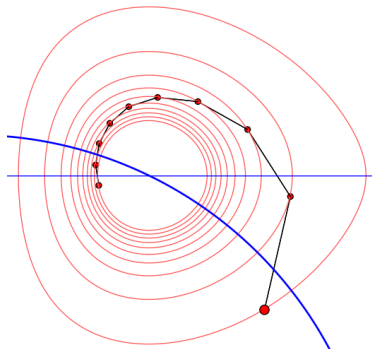
- Borwein and Sims (2011) conjectured global, proved local
- Aragón Artacho and Borwein (2013) provided partial answers
- Borwein (2016) furnished images similar to right



04: Lyapunov Functions

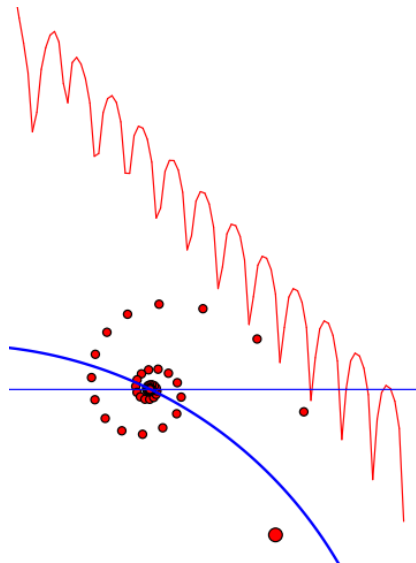
Case study: sphere/line DR:

- Benoist (2015) constructed Lyapunov function
- Borwein et al. (2018) showed global results don't extend to sphere generalizations
- Dao and Tam (2019); Giladi and Rüffer (2019) extended Benoist's framework to more general graphs locally
- Asymptotic stability essentially guarantees existence of such functions (Kellett and Teel, 2005; Giladi and Rüffer, 2019; Lindstrom, 2020)



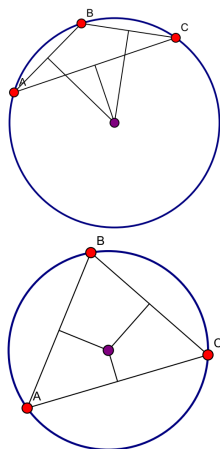
05: Oscillations

- When spiraling, plotting the changes in projections onto line (shadow sequence) produces smooth arches and sharp valleys
- Same pattern observed in many contexts. Discussed in (Lamichhane et al., 2019).



06: Circumcenter Construction

- The circumcenter of a 3-tuple (A, B, C) is the center of a circle that contains A, B, C .
- It lies at the intersection of the perpendicular bisectors of triangle ABC
- Exists finitely whenever A, B, C not both distinct and colinear.



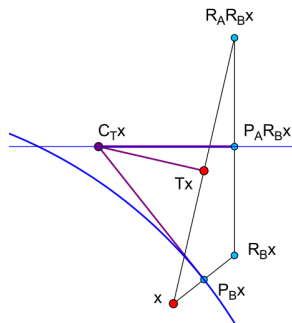
07: Circumcentering Reflections Method (CRM)

- Behling et al. (2018b,a) introduced CRM: $C_T x = C(x, R_A x, R_A R_B x)$.
- Bauschke et al. (2018a,b): sufficient conditions guarantee *properness* of operator (existence of updates)
- Dizon et al. (2019) introduced GCRM:

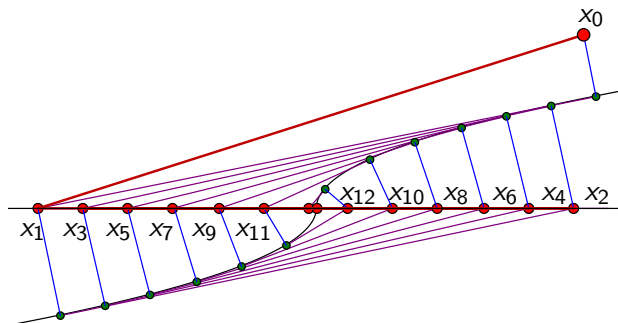
$$C_T : x \mapsto \begin{cases} T_{A,B}x & \text{if colinear} \\ C_T x & \text{otherwise} \end{cases}.$$

Also used subspace invariance to show local quadratic convergence with plane curves & lines.

- Behling et al. (2019): convex, product space formulation global convergence

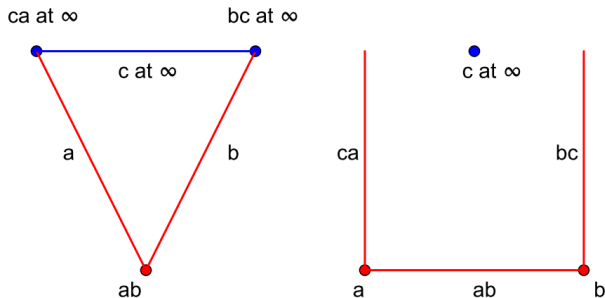


08: Newton–Raphson characterization



- Perpendicular bisectors of reflections across hyperplanes are the hyperplanes
- Perpendicular bisectors of reflections across curves are tangents
- When one set is a hyperplane and one set is a graph, CRM “resembles” Newton–Raphson (Dizon et al., 2019).

09: Duality



- Renaissance geometry: projective duality of lines and points.
- Convex optimization: Fenchel–Moreau–Rockafellar duality of epigraphs of convex functions.
- Duality of algorithms Gabay (1983):
 - ADMM: $\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + g(z) \quad \text{s.t.} \quad Mx = z \in \mathbb{R}^m.$
 - DR: $\underset{\lambda \in \mathbb{R}^m}{\text{minimize}} \quad f^* \circ (-M^T)(\lambda) + g^*(\lambda)$

10: ADMM and Basis Pursuit

Basis pursuit:

minimize $\|x\|_1$ s.t. $Ax = b$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{\nu \times n}$, $b \in \mathbb{R}^\nu$, $\nu < n$.

ADMM:

$$f := \iota_S, \quad S := \{x \in \mathbb{R}^n \mid Ax = b\},$$

$$M := \text{Id}, \quad g : z \rightarrow \|z\|_1, \quad E, Y := \mathbb{R}^n.$$

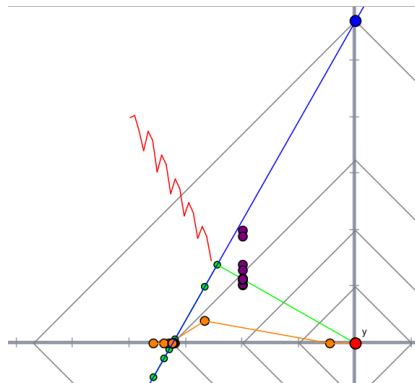
$$x_{k+1} \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \{f(x) + g(z_k)\}$$

$$+ \langle \lambda_k, Mx - z_k \rangle + \frac{c}{2} \|Mx - z\|^2\}$$

$$z_{k+1} \in \underset{z \in \mathbb{R}^m}{\operatorname{argmin}} \{f(x_{k+1}) + g(z)\}$$

$$+ \langle \lambda_k, Mx_{k+1} - z \rangle + \frac{c}{2} \|Mx_{k+1} - z\|^2\}$$

$$\lambda_{k+1} = \lambda_k + c(Mx_{k+1} - z_{k+1}).$$



11: Primal/Dual Algorithms

Set: $d_1 := f^* \circ (-M^T)$, $d_2 := g^*$

Primal reconstruction:

$$cMx_{k+1} = \text{prox}_{cd_1}(R_{cd_2}y_k) - R_{cd_2}y_k$$

$$cz_k = y_k - \text{prox}_{cd_2}y_k$$

$$\lambda_k = \text{prox}_{cd_2}y_k$$

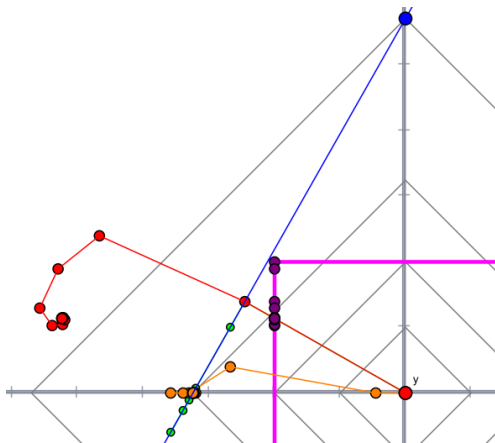
Dual reconstruction:

$$y_k = \lambda_k + cz_k$$

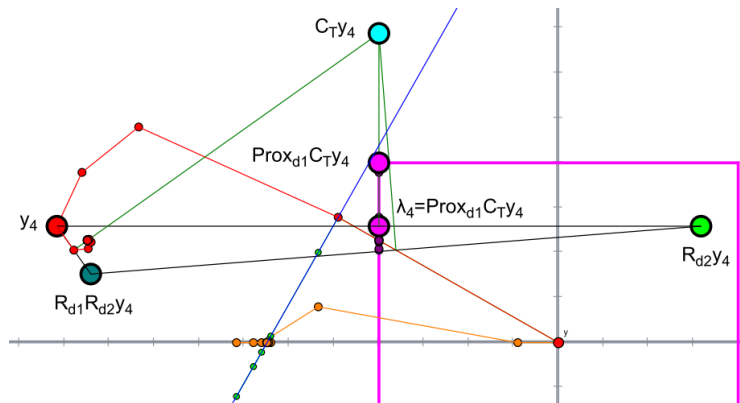
$$R_{cd_2}y_k = \lambda_k - cz_k$$

$$R_{cd_1}R_{cd_2}y_k = \lambda_k - cz_k + 2cMx_{k+1}$$

See (Eckstein and Yao,
2015).



12: Failure of CRM for primal/dual

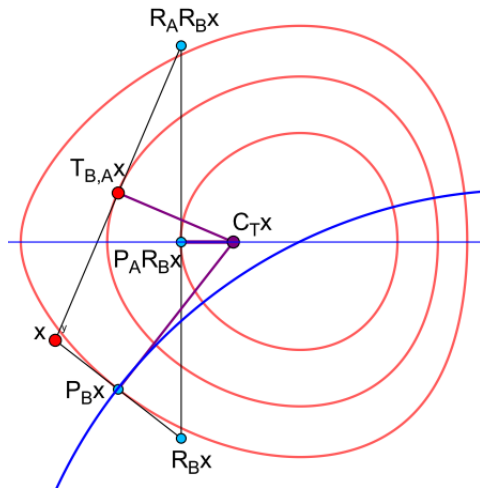


For basis pursuit, $\text{prox}_{d_2} = \text{proj}_{B_\infty}$, so the perpendicular bisector of $(y_k, R_{cd2}y_k)$ contains a face of B_∞ , so CRM generically fails.

13: CRM and Lyapunov functions

Theorem 1 (Lindstrom (2020))

For Lyapunov functions V from Benoist (2015); Dao and Tam (2019), CRM updates lie on the intersections of hyperplanes that contain subgradient descent trajectories of V from the points P_{B^X} , $P_{A^X}R_{B^X}$, and T_{B,A^X} . If the set order is reversed, the analogous result holds with the points P_{A^X} , $P_{B^X}R_{A^X}$, and T_{A,B^X} .

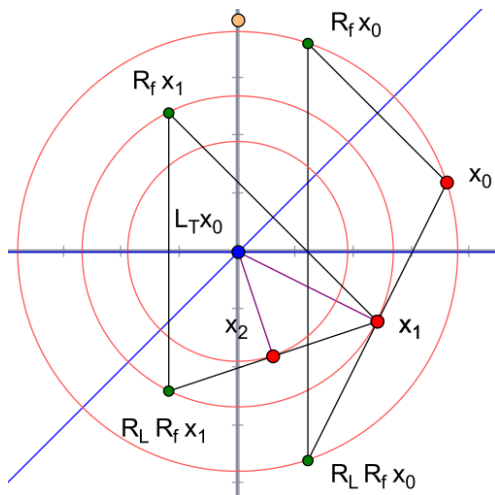


15: An interesting example

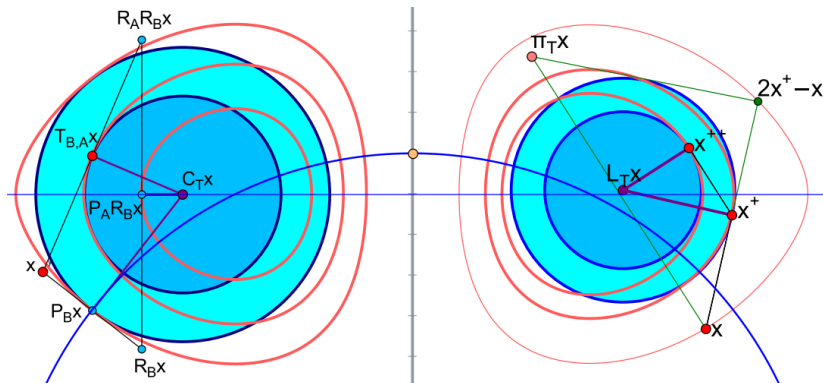
Proposition 1 (Lindstrom (2020))

When $T_{A,B}$ is the Douglas–Rachford projection operator and A, B are lines, $L_{T_{A,B}}x$ is a fixed point for any x .

- Easy to prove: all subgradient descent trajectories from all points intersect in the fixed set
- Another view: any quadratic surrogate that matches gradients of V at the sample points will be “equal” to V



16: Quadratic surrogate characterization



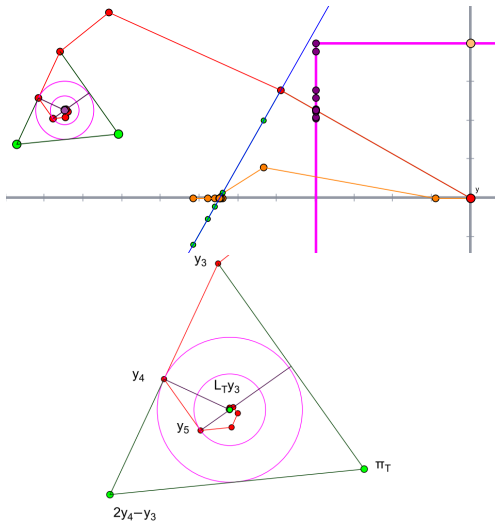
Theorem 3 (Lindstrom (2020))

Under appropriate assumptions, L_T and C_T return minimizers of quadratic surrogates for Lyapunov functions that describe the convergence of algorithms admitted by iteratively applying the operator T (where T is Douglas–Rachford specifically in the C_T case).

17: Primal/dual centering

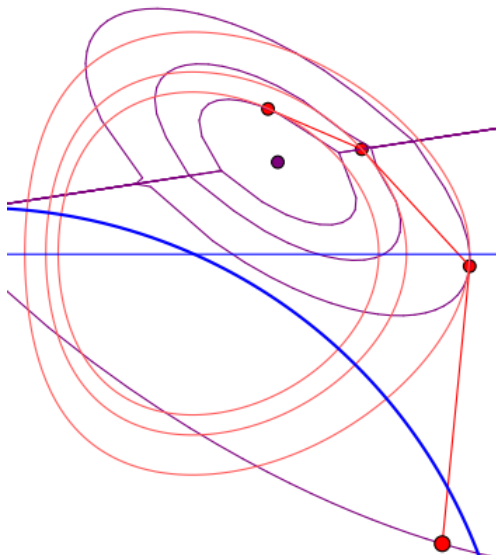
Primal/dual acceleration
of ADMM/DR:

1. Compute primal, reconstruct dual
2. Apply L_T to dual
3. Propagate centered step update back to ADMM variables
4. Check objective function against improvement from regular update, pick winning candidate and repeat

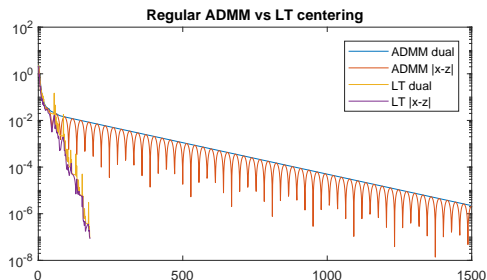


18: More general quadratic surrogate minimization

1. One could also consider a spherical surrogate built from more points (higher dimensions)...
2. or adaptive algorithms that compare and choose from among surrogate-offered update candidates...
3. Or a more general elliptical (not just spherical) surrogate...



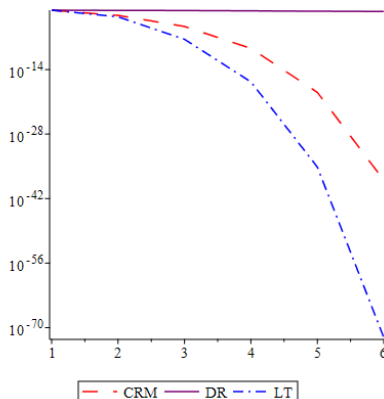
Computational Evidence: Basis Pursuit



For 1,000 problems of dimension 300, L_T solved every problem:

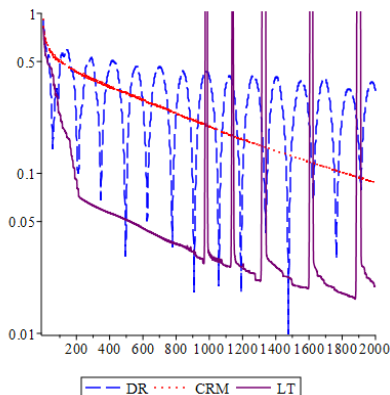
	wins	min	Q1	median	Q3	max
vanilla ADMM	1	278	995	1582	2932	761,282
L_T -centering	999	87	165	221	324	94,591

Computational Evidence: Sphere and Affine Subspace



- With A and sphere and B a line and $T_{A,B} = (R_B R_A + \text{Id})/2$, the iterated composition $L_T \circ P_B$ exhibits quadratic convergence rate (not yet proven)
- Neil Dizon, Jeffrey Hogan, and SBL, “Circumcentering Reflections for Nonconvex Feasibility Problems,” (2020), arXiv:1910.04384.

Computational Evidence: Solving a boundary value ODE



1. Neil Dizon, Jeffrey Hogan, and SBL, "Circumcentering Reflections for Nonconvex Feasibility Problems," (2020), arXiv:1910.04384.
2. Bishnu P. Lamichhane, SBL, and Brailey Sims, "Application of Projection Algorithms to Differential Equations: Boundary Value Problems," ANZIAM, (2019) 61(1), 23-46.

Computational Evidence: Finding Wavelets

	Algorithm	cases solved	solved by all	when solved by all				
				wins	Q1	mean	Q3	median
Problem 1	DR	51	51	0	194	211	215	201
	GCRM	51	51	13	36	28	40	38
	L_T	51	51	38	29	36	39	33
Problem 2	DR	96	79	0	176	182	186	185
	GCRM	79	79	22	31	33	35	33
	L_T	96	79	57	28	32	33	31

Table: Performance during stage 2 of a 2-stage search.

- Neil Dizon, Jeffrey Hogan, and SBL, “Centering Projection Methods for Wavelet Feasibility Problems,” ISAAC 2019: The 30th International Symposium on Algorithms and Computation proceeding volume Current Trends in Analysis, its Applications and Computation, P. Cerejeiras, M. Reissig, I. Sabadini, J. Toft (Editors), in series Birkhäuser series Research Perspectives. arXiv:2005.05687 (2020).

Summary

Theoretical contributions:

- CRM: Lyapunov characterization (including nonconvex cases)
- New method L_T :
 1. problem agnostic: generically implementable
 2. *operator* agnostic: ADMM, DR, FISTA, Nesterov...
 3. does not require subproblems: black box compatible
 4. satisfies Lyapunov properties with remarkably few structural assumptions: primal/dual adaptable
 5. not convexity reliant: good candidate for online optimization
- Both methods: quadratic surrogacy characterization

General lessons:

- Use tools like *Cinderella* and *Geogebra*
- Excursions into the weeds (e.g. highly specific problems/structure) yield insights about more general problems
- Look for bridges between seemingly disparate areas of research (e.g. Lyapunov functions, centering)

Thank you!

L., “Computable centering methods for spiraling algorithms and their duals, with motivations from the theory of Lyapunov functions.” arXiv preprint arXiv:2001.10784, (2020).

Please consider:

- Razom for Ukraine
- World Central Kitchen

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- Jonathan M. Borwein, Scott B. Lindstrom, Brailey Sims, Matthew Skerritt, and Anna Schneider. Dynamics of the Douglas–Rachford method for ellipses and p-spheres. *Set-Valued Anal.*, 26(2):385–403, 2018.
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- Ohad Giladi and Björn S Rüffer. A lyapunov function construction for a non-convex douglas–rachford iteration. *Journal of Optimization Theory and Applications*, 180(3):729–750, 2019.
- Christopher M Kellett and Andrew R Teel. On the robustness of ℓ_1 -stability for difference inclusions: Smooth discrete-time Lyapunov functions. *SIAM Journal on Control and Optimization*, 44(3):777–800, 2005.
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References V

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