Roots of Identity Operator and Proximal Mappings: Classical and Phantom Cycle and Gap Vectors

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Abstract

Using the Attouch-Théra Duality, we study the cycles, gap vectors of compositions of proximal mappings.

1. Sufficient conditions are given under which the cycles and gap vectors exist.
2. Phantom cycles and gap vectors are introduced to tackle the situations when the classical ones do not exist.

Recently, Simons provided a lemma for a support function of a closed convex set to study the geometry conjecture on cycles of projections. We

1. extend Simons's lemma to closed convex functions,
2. show its connections to Attouch–Théra duality, and
3. use it to characterize classical and phantom cycles and gap vectors.

One can study phantom cycles and gap vectors of a convex function associated with an arbitrary isometry, rather than just the right-shift operator.

Outline

1. What is a cycle for a composition of proximal mappings?
2. Classical cylcles and gap vectors via the Attouch-Théra duality
3. Imagination: phantom cycle and gap vectors
4. Examples
5. Simons: mth roots of identity operator and an average operator
6. Characterizations of classical cycle and gap vectors
7. Phantom cycle and gap vectors for arbitrary isometry R
8. Conclusions

Outline

Setup

X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle\colon X\times X\to[0,+\infty[$ and induced norm $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$.

The set of proper lower semicontinuous convex functions from X to $(-\infty,+\infty]$ is denoted by $\Gamma_0(X)$.

In the product space $\mathbf{X}=X^m$ with $m\in\mathbb{N}$, we let

$$\Delta=\big\{(x,\ldots,x)\mid x\in X\big\},$$

$$\mathbf{R}:\mathbf{X}\to\mathbf{X}:(x_1,x_2,\ldots,x_m)\mapsto(x_m,x_1,\ldots,x_{m-1}),\ \text{and}$$

$$\|\mathbf{x}\|=\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}=\sqrt{\|x_1\|^2+\cdots+\|x_m\|^2}$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_m)$. For a finite family of functions $(f_i)_{i=1}^m$ in $\Gamma_0(X)$, define its separable sum by

$$\mathbf{f}=f_1\oplus\cdots\oplus f_m:\mathbf{X}\to\ ]-\infty,+\infty]:(x_1,\ldots,x_m)\mapsto\sum_{i=1}^m f_i(x_i). \tag{1}$$

The proximal mapping of f_i is defined by $\mathrm{prox}_{f_i} = (\mathrm{Id} + \partial f_i)^{-1}$ where ∂f_i denotes the subdifferential of f_i.

A *cycle* of **f** is a vector $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbf{X}$ such that

$$z_1 = \mathrm{prox}_{f_1} z_m, \quad z_2 = \mathrm{prox}_{f_2} z_1, \quad z_3 = \mathrm{prox}_{f_3} z_2, \cdots, \tag{2}$$

$$z_{m-1} = \mathrm{prox}_{f_{m-1}} z_{m-2}, \quad z_m = \mathrm{prox}_{f_m} z_{m-1}. \tag{3}$$

The set of all cycles of **f** will be denoted by **Z**.

In the frame work of product space **X**, with $\mathbf{z} = (z_1, \ldots, z_m)$, the operator form of (2)–(3) is

$$\mathbf{z} = \mathrm{prox}_{\mathbf{f}} \mathbf{R}\mathbf{z}, \text{ equivalently,} \tag{4}$$

in terms of monotone operators

$$0 \in \partial \mathbf{f}(\mathbf{z}) + \mathbf{z} - \mathbf{R}\mathbf{z}, \tag{5}$$

where the displacement mapping $\mathrm{Id} - \mathbf{R}$ is maximally monotone but not a gradient of convex function unless $m = 2$.

Notation

The Fenchel conjugate of f is

$$f^* : X \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X}(\langle x, x^* \rangle - f(x)).$$

The infimal convolution of f, g is

$$f \Box g : X \to [-\infty, +\infty] : x \mapsto \inf_{y \in X}(f(y) + g(x - y)),$$

and it is exact at a point $x \in X$ if $(\exists y \in X)\ (f \Box g)(x) = f(y) + g(x - y)$; $f \Box g$ is exact if it is exact at every point of its domain.

The subdifferential of f is the set-valued operator

$$\partial f : X \rightrightarrows X : x \mapsto \big\{x^* \in X \mid (\forall y \in X)\ f(y) \geq f(x) + \langle u, y - x \rangle\big\}.$$

We use $\operatorname{cl} f$ for the lower semicontinuous hull of f.

For a set $C \subset X$, its indicator function is defined by

$$\iota_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

When the set C is nonempty closed convex, we write $P_C = \operatorname{prox}_{\iota_C}$ for the projection operator and $N_C = \partial \iota_C$ for the normal cone.

Let $\operatorname{Id} : X \to X$ be the identity operator. An operator $N : X \to X$ is

1. nonexpansive if $(\forall x, y \in X)\ \|Nx - Ny\| \leq \|x - y\|$;
2. firmly nonexpansive if $2N - \operatorname{Id}$ is nonexpansive;
3. β-cocercive if βN is firmly nonexpansive for some $\beta \in \,]0, +\infty[$.

Prime examples of firmly nonexpansive mappings are proximal mappings of convex functions.

As usual, $\operatorname{Fix} N = \{x \in X \mid Nx = x\}$ denotes the set of fixed points of N.

For a monotone operator $A : X \rightrightarrows X$, we write $\widetilde{A} = (-\operatorname{Id}) \circ A^{-1} \circ (-\operatorname{Id})$.

Blanket assumptions

Recall the diagonal set in X^m by

$$\Delta = \big\{(x, \ldots, x) \mid x \in X\big\}.$$

Throughout, we shall assume that

1. $(f_i)_{i=1}^m$ are in $\Gamma_0(X)$, and $\mathbf{f}$ is given by (1).
2. $$\operatorname{dom}(\mathbf{f}^* + \iota_\Delta^*) = \operatorname{dom}(\mathbf{f}^* + \iota_{\Delta^\perp}) \neq \varnothing, \tag{6}$$
   equivalently, $\operatorname{dom}\mathbf{f}^* \cap \Delta^\perp \neq \varnothing$. This will assure that $\mathbf{f}\Box\iota_\Delta$ is proper convex, and possess a continuous minorant.

Some facts

The key tool we shall use is the following Attouch-Théra duality.

Fact 1 (Attouch-Théra duality [3])

Let $A, B : X \rightrightarrows X$ be maximally monotone operators. Let S be the solution set of the primal problem

$$\text{find } x \in X \text{ such that } 0 \in Ax + Bx. \tag{7}$$

Let S^* be the solution set of the dual problem

$$\text{find } x^* \in X \text{ such that } 0 \in A^{-1}x^* + \widetilde{B}(x^*). \tag{8}$$

Then

1. $S = \{x \in X \mid (\exists\, x^* \in S^*)\ x^* \in Ax \text{ and } -x^* \in Bx\}$.
2. $S^* = \{x^* \in X \mid (\exists\, x \in S)\ x \in A^{-1}x^* \text{ and } -x \in \widetilde{B}(x^*)\}$.

Important properties of the circular right shift operator come as follows.

Fact 2

For the circular right shift operator $\mathbf{R}$, *the following hold:*

1. $\operatorname{Id} - \mathbf{R}$ *is maximally monotone.*
2. $(\operatorname{Id} - \mathbf{R})^{-1} = \frac{1}{2}\operatorname{Id} + N_{\Delta^\perp} + T$ *where* $T : \mathbf{X} \to \mathbf{X}$ *is a skew operator defined by*
$$T = \frac{1}{2m}\sum_{k=1}^{m-1}(m-2k)\mathbf{R}^k.$$
In particular, $\operatorname{dom}(\operatorname{Id} - \mathbf{R})^{-1} = \Delta^\perp$*.*
3. $(\frac{1}{2}\operatorname{Id} + T)^{-1} = \operatorname{Id} - \mathbf{R} + 2P_\Delta$*.*

Lemma 3

Let $f : X \to \,]-\infty, +\infty]$ be proper and convex, and $x \in X$. Then the following hold:

1. *If $\partial f(x) \neq \varnothing$, then f is lower semicontinuous at x.*
2. *If $f(x) = \operatorname{cl} f(x)$, that is, f is lower semicontinuous at x, then $\partial f(x) = \partial \operatorname{cl} f(x)$.*
3. *In general, $\partial f \subseteq \partial \operatorname{cl} f$.*

Lemma 4

Let $f, g \in \Gamma_0(X)$ and $x, y \in X$. Then the following hold:

1. *If $(f \Box g)(x) = f(y) + g(x-y)$, then $\partial(f \Box g)(x) = \partial f(y) \cap \partial g(x-y)$.*
2. *If $\partial f(y) \cap \partial g(x-y) \neq \varnothing$, then $(f \Box g)(x) = f(y) + g(x-y)$ and*
$$\partial(f \Box g)(x) = \partial f(y) \cap \partial g(x-y).$$
3. *In general, $\partial(f \Box g)(x) \supseteq \partial f(y) \cap \partial g(x-y)$.*

Fact 5

Suppose that $S = \bigcap_{i=1}^{m} \operatorname{argmin} f_i \neq \varnothing$. Then
$$\mathbf{Z} = \big\{(z, \dots, z) \mid z \in S\big\}.$$

Outline

Using the Attouch-Théra duality with $A = \partial \mathbf{f}$ and $B = \operatorname{Id} - \mathbf{R}$, and the identity

$$-\operatorname{Id} \circ (\operatorname{Id} - \mathbf{R})^{-1} \circ (-\operatorname{Id}) = (\operatorname{Id} - \mathbf{R})^{-1}$$

for linear relation $(\operatorname{Id} - \mathbf{R})^{-1}$, we can formulate the primal-dual inclusion problem:

$$(P) \quad 0 \in \partial \mathbf{f}(\mathbf{x}) + (\operatorname{Id} - \mathbf{R})\mathbf{x}, \tag{9}$$

$$(D) \quad 0 \in (\partial \mathbf{f})^{-1}(\mathbf{y}) + (\operatorname{Id} - \mathbf{R})^{-1}\mathbf{y}. \tag{10}$$

Theorem 6

The solution set of (D) is at most a singleton (possibly empty).

Proof.

Since $(\operatorname{Id} - \mathbf{R})^{-1} = \frac{1}{2}\operatorname{Id} + N_{\Delta^\perp} + T$ by Fact 2, the monotone operator

$$\partial \mathbf{f}^{-1} + (\operatorname{Id} - \mathbf{R})^{-1} = \frac{1}{2}\operatorname{Id} + (N_{\Delta^\perp} + T + \partial \mathbf{f}^{-1})$$

is strongly monotone, so $[\partial \mathbf{f}^{-1} + (\operatorname{Id} - \mathbf{R})^{-1}]^{-1}(0)$ is at most a singleton. □

Theorem 7

Consider the sets of classical cycles and classical gap vectors defined respectively by

$$\mathbf{Z} = \left\{\mathbf{x} \in \mathbf{X} \mid 0 \in \partial\mathbf{f}(\mathbf{x}) + (\mathrm{Id} - \mathbf{R})\mathbf{x}\right\}, \tag{11}$$

$$\mathbf{G} = \left\{\mathbf{y} \in \mathbf{X} \mid 0 \in (\partial\mathbf{f})^{-1}(\mathbf{y}) + (\mathrm{Id} - \mathbf{R})^{-1}\mathbf{y}\right\}. \tag{12}$$

We have

1. $\mathbf{Z} = \bigcup_{\mathbf{y}\in\mathbf{G}}(\mathrm{Id} - \mathbf{R})^{-1}(-\mathbf{y}) \cap (\partial\mathbf{f})^{-1}(\mathbf{y})$.
2. $\mathbf{G} = \bigcup\left\{\mathbf{R}\mathbf{x} - \mathbf{x} \mid \mathbf{x} \in \mathbf{Z}\right\}$. *If* $\mathbf{G} \neq \varnothing$, *then* $\mathbf{G}$ *is a singleton* $\mathbf{y} \in \Delta^{\perp}$ *and* $\mathbf{y} = \mathbf{R}\mathbf{x} - \mathbf{x}$ *for every* $\mathbf{x} \in \mathbf{Z}$.

Outline

Extending the dual approach

Since the linear relation $(\operatorname{Id}-\mathbf{R})^{-1}=\frac{1}{2}\operatorname{Id}+N_{\Delta^\perp}+T$ by Fact 2, and $\partial\iota_\Delta^*=\partial\iota_{\Delta^\perp}=N_{\Delta^\perp}$, we have

$$\partial\mathbf{f}^{-1}+(\operatorname{Id}-\mathbf{R})^{-1}=\partial\mathbf{f}^*+\frac{1}{2}\operatorname{Id}+T+\partial\iota_\Delta^*=\partial\mathbf{f}^*+\partial\iota_\Delta^*+\frac{1}{2}\operatorname{Id}+T \tag{13}$$

$$\subseteq\partial(\mathbf{f}^*+\iota_\Delta^*)+\frac{1}{2}\operatorname{Id}+T \tag{14}$$

$$=\frac{1}{2}\big[\operatorname{Id}+\big(2T+2\partial(\mathbf{f}^*+\iota_\Delta^*)\big)\big]. \tag{15}$$

The enlarged dual

$$(\tilde{D})\quad 0\in\partial(\mathbf{f}^*+\iota_\Delta^*)(\mathbf{y})+\frac{1}{2}\mathbf{y}+T\mathbf{y} \tag{16}$$

always has a unique solution. We call the $\mathbf{y}$ given by (16) as the *phantom gap vector* of $\operatorname{cl}(\mathbf{f}\Box\iota_\Delta)$.

Extending the primal approach

One can also start from the primal

$$(P) \quad 0 \in \partial \mathbf{f}(\mathbf{u}) + (\mathrm{Id} - \mathbf{R})\mathbf{u}.$$

Because $\partial \mathbf{f} + (\mathrm{Id} - \mathbf{R})$ is already maximally monotone by [4], one cannot do enlargements so that (P) has a solution. We need to rewrite it in an equivalent form. In view of

$$-(\mathrm{Id} - \mathbf{R})\mathbf{u} \in \partial \mathbf{f}(\mathbf{u}), \quad -(\mathrm{Id} - \mathbf{R})\mathbf{u} \in \Delta^{\perp},$$

Lemmas 4 and 3, we have

$$\begin{aligned} -(\mathrm{Id} - \mathbf{R})\mathbf{u} &\in \partial \mathbf{f}(\mathbf{u}) \cap \Delta^{\perp} && (17) \\ &= \partial \mathbf{f}(\mathbf{u}) \cap \partial \iota_{\Delta}(\mathbf{d}) \subseteq \partial(\mathbf{f} \Box \iota_{\Delta})(\mathbf{u} + \mathbf{d}) && (18) \\ &\subseteq \partial[\mathrm{cl}\,(\mathbf{f} \Box \iota_{\Delta})](\mathbf{u} + \mathbf{d}), && (19) \end{aligned}$$

where $\mathbf{d} \in \Delta$.

Because $(\operatorname{Id} - \mathbf{R})(\mathbf{d}) = 0$, we can write equations (17)–(19) as

$$0 \in \partial[\operatorname{cl}(\mathbf{f} \Box \iota_\Delta)](\mathbf{u} + \mathbf{d}) + (\operatorname{Id} - \mathbf{R})(\mathbf{u} + \mathbf{d}).$$

With

$$\mathbf{d} = (-\sum_{i=1}^{m} u_i/m, \ldots, -\sum_{i=1}^{m} u_i/m) \in \Delta$$

and

$$\mathbf{x} = \mathbf{u} + \mathbf{d} \in \Delta^\perp,$$

we have

$$0 \in \partial[\operatorname{cl}(\mathbf{f} \Box \iota_\Delta)](\mathbf{x}) + (\operatorname{Id} - \mathbf{R})(\mathbf{x}), \text{ and } \mathbf{x} \in \Delta^\perp. \tag{20}$$

The solution $\mathbf{x}$ given by (20) is called a *phantom cycle* of $\operatorname{cl}(\mathbf{f} \Box \iota_\Delta)$.

The primal-dual approach

The phantom cycle and gap vectors of $\mathrm{cl}(\mathbf{f}\Box\iota_{\Delta})$ can be put into the frame work of the Attouch-Théra duality.

Theorem 8

Consider the following Attouch-Théra primal-dual problems

$$(\tilde{P}) \quad 0 \in \partial[\mathrm{cl}(\mathbf{f}\Box\iota_{\Delta})](\mathbf{x}) + (\mathrm{Id} - \mathbf{R})\mathbf{x} \text{ and } \mathbf{x} \in \Delta^{\perp}, \tag{21}$$

$$(\tilde{D}) \quad 0 \in \partial(\mathbf{f}^{*} + \iota_{\Delta}^{*})(\mathbf{y}) + \frac{1}{2}\mathbf{y} + T\mathbf{y}. \tag{22}$$

Then the following hold:

1. *$(\tilde{D})$ is the Attouch-Théra dual of $(\tilde{P})$, and $(\tilde{D})$ has a unique solution.*
2. *$(\tilde{P})$ has a unique solution.*

Lemma 9

We have $\operatorname{ran}\partial[\operatorname{cl}(\mathbf{f}\Box\iota_\Delta)] \subseteq \Delta^\perp$.

- For $A : X \rightrightarrows X$, $\operatorname{ran} A$ denotes the range of A.

Proof.

①: Let us consider the Attouch-Théra dual of $(\tilde{D})$. As $(\mathbf{f}^* + \iota_\Delta^*)^* = \mathrm{cl}(\mathbf{f}\Box\iota_\Delta)$, we have

$$0 \in \partial[\mathrm{cl}(\mathbf{f}\Box\iota_\Delta)](\mathbf{x}) + \left(\frac{1}{2}\operatorname{Id} + T\right)^{-1}(\mathbf{x}). \tag{23}$$

Since $(\frac{1}{2}\operatorname{Id} + T)^{-1} = \operatorname{Id} - \mathbf{R} + 2P_\Delta$ by Fact 23, we obtain

$$0 \in \partial[\mathrm{cl}(\mathbf{f}\Box\iota_\Delta)](\mathbf{x}) + (\operatorname{Id} - \mathbf{R})\mathbf{x} + 2P_\Delta(\mathbf{x}).$$

Because $\operatorname{ran}(\operatorname{Id} - \mathbf{R}) \subseteq \Delta^\perp$, and Lemma 9, the above implies

$$-2P_\Delta(\mathbf{x}) \in \partial[\mathrm{cl}(\mathbf{f}\Box\iota_\Delta)](\mathbf{x}) + (\operatorname{Id} - \mathbf{R})\mathbf{x}$$

from which $2P_\Delta(\mathbf{x}) \in \Delta \cap \Delta^\perp$, so $P_\Delta(\mathbf{x}) = 0$, and $\mathbf{x} \in \Delta^\perp$. Hence, (23) is equivalent to

$$0 \in \partial[\mathrm{cl}(\mathbf{f}\Box\iota_\Delta)](\mathbf{x}) + (\operatorname{Id} - \mathbf{R})\mathbf{x}, \text{ and } \mathbf{x} \in \Delta^\perp, \tag{24}$$

which is precisely (21).
$(\tilde{D})$ has a unique solution by Theorem 6. □

Recovering the classical cycle from the phantom cycle under ...

Theorem 10

Let $\mathbf{x}$ *be a phantom cycle of* $\operatorname{cl}(\mathbf{f}\Box\iota_\Delta)$*, i.e.,*

$$(\tilde{P}) \quad 0 \in \partial[\operatorname{cl}(\mathbf{f}\Box\iota_\Delta)](\mathbf{x}) + (\operatorname{Id} - \mathbf{R})\mathbf{x}, \text{ and } \mathbf{x} \in \Delta^\perp.$$

If

$$\operatorname{cl}(\mathbf{f}\Box\iota_\Delta)(\mathbf{x}) = (\mathbf{f}\Box\iota_\Delta)(\mathbf{x}), \text{ and } \mathbf{f}\Box\iota_\Delta \text{ is exact at } \mathbf{x},$$

then $\mathbf{x} = \mathbf{u} + \mathbf{v}$, $\mathbf{v} = (-\sum_{i=1}^m u_i/m, \ldots, -\sum_{i=1}^m u_i/m) \in \Delta$*, and*

$$0 \in \partial\mathbf{f}(\mathbf{u}) + (\operatorname{Id} - \mathbf{R})\mathbf{u}.$$

Consequently, $\mathbf{u}$ *is a classical cycle for* $\mathbf{f}$*.*

Classical cycles become the phantom cycle under a shift

Theorem 11

Let $\mathbf{u} = (u_1, \ldots, u_m)$ *with* $u_i \in X$ *for* $i = 1, \ldots, m$, *and let* $\mathbf{u}$ *be a classical cycle for* $\mathbf{f}$, *i.e.,*

$$0 \in \partial \mathbf{f}(\mathbf{u}) + (\mathrm{Id} - \mathbf{R})\mathbf{u}. \tag{25}$$

Set $\mathbf{v} = (-\sum_{i=1}^m u_i/m, \ldots, -\sum_{i=1}^m u_i/m) \in \Delta$ *and* $\mathbf{x} = \mathbf{u} + \mathbf{v}$. *Then*

1. $\mathbf{f} \Box \iota_\Delta$ *is lower semicontinuous and exact at* $\mathbf{x}$.
2. $\mathbf{x} \in \Delta^\perp$ *and* $\mathbf{x}$ *solves*

$$(\tilde{P}) \quad 0 \in \partial(\mathbf{f} \Box \iota_\Delta)(\mathbf{x}) + (\mathrm{Id} - \mathbf{R})\mathbf{x} = \partial[\mathrm{cl}(\mathbf{f} \Box \iota_\Delta)](\mathbf{x}) + (\mathrm{Id} - \mathbf{R})\mathbf{x}. \tag{26}$$

Consequently, $\mathbf{x}$ *is a phantom cycle for* $\mathrm{cl}(\mathbf{f} \Box \iota_\Delta)$.

The following result summarizes the relationship among the classical cycles, phantom cycle and gap vectors.

Corollary 12

With $\mathbf{x}$ *and* $\mathbf{y}$ *given in Theorem 8, the following hold:*

1. $\mathbf{x}, \mathbf{y} \in \Delta^{\perp}$.
2. $\mathbf{y} = \mathbf{R}\mathbf{x} - \mathbf{x}$.
3. $\mathbf{x} = -\frac{\mathbf{y}}{2} - T\mathbf{y}$.
4. $\mathbf{Z} = (\mathbf{x} + \Delta) \cap (\partial \mathbf{f})^{-1}(\mathbf{R}\mathbf{x} - \mathbf{x}) = (\operatorname{Id} - \mathbf{R})^{-1}(-\mathbf{y}) \cap (\partial f)^{-1}(\mathbf{y})$.
5. $\mathbf{Z} \subseteq (F_1 \times \cdots \times F_m) \cap (\operatorname{Id} - \mathbf{R})^{-1}(-\mathbf{y})$.

Characterization of $\mathbf{Z} \neq \varnothing$ via phantom cycles

Recall the parallel sum $(\forall x \in X)\ (\partial f \Box \partial g)(x) = \bigcup_{x=u+v} \partial f(u) \cap \partial g(v)$.

Corollary 13

Let $\mathbf{x}$ *by given in Theorem 8. Then the following are equivalent:*

1. $\mathbf{Z} \neq \varnothing$.
2. $\mathrm{cl}(\mathbf{f} \Box \iota_\Delta)(\mathbf{x}) = (\mathbf{f} \Box \iota_\Delta)(\mathbf{x})$ *and* $\mathbf{f} \Box \iota_\Delta$ *is exact at* $\mathbf{x}$.
3. $(\partial \mathbf{f} \Box \partial \iota_\Delta)(\mathbf{x}) \neq \varnothing$.

Outline

For $C_i \subseteq X$, $i = 1, \dots, m$, we let

$$\mathbf{C} = C_1 \times \cdots \times C_m \subseteq \mathbf{X}.$$

The first example illustrates the concepts of generalized cycles and gap vectors when the classical ones do not exist.

Example 14

Let $\alpha \geq 0$. Consider

$$C_1 = \operatorname{epi} \exp = \big\{(x, r) \mid r \geq \exp(-x) + \alpha \text{ and } x \in \mathbb{R}\big\}, \text{ and } C_2 = \mathbb{R} \times \{0\}.$$

Then

1. $\iota_{\mathbf{C}}$ has neither a cycle nor a gap vector.
2. $\operatorname{cl}(\iota_{\mathbf{C}} \Box \iota_{\Delta}) = \iota_{\overline{\mathbf{C}+\Delta}}$ has both a generalized cycle and a generalized gap vector, namely $\mathbf{x} = (0, \alpha/2, 0, -\alpha/2), \mathbf{y} = (0, -\alpha, 0, \alpha) \in \mathbb{R}^4$.

The second example characterizes when the set of cycles is a singleton or infinite for a finite number of lines in a Hilbert space.

Example 15

Given m sets in X: $C_i = \left\{a_i + t_i b_i \mid t_i \in \mathbb{R}\right\}$ where $a_i \in X$ and $b_i \in X \setminus \{0\}$ for $i = 1, \ldots, m$. Then the following hold:

1. $\iota_{\mathbf{C}}$ always has a classical cycle, i.e., $\mathbf{Z} \neq \varnothing$.
2. $\iota_{\mathbf{C}}$ has a unique classical cycle if and only if the set of vectors $\left\{b_i \mid i = 1, \ldots, m\right\}$ is not parallel.
3. $\iota_{\mathbf{C}}$ has infinitely many classical cycles if and only if the set of vectors $\left\{b_i \mid i = 1, \ldots, m\right\}$ is parallel.

Finding phantom cycle and gap vectors

In view of the possibility of $\mathbf{Z} = \varnothing$, one can consider the extended Attouch-Théra primal-dual:

$$\text{(EP)} \quad 0 \in \partial\, \mathrm{cl}(\mathbf{f} \Box \iota_{\Delta})(\mathbf{x}) + (\mathrm{Id} - \mathbf{R})\mathbf{x}, \tag{27}$$

$$\text{(ED)} \quad 0 \in \partial(\mathbf{f}^* + \iota_{\Delta}^*)(\mathbf{y}) + \frac{1}{2}\mathbf{y} + T\mathbf{y}. \tag{28}$$

Both (EP) and (ED) always have solutions.

While (EP) gives all generalized cycles, (ED) gives the unique generalized gap vector for $\mathrm{cl}(\mathbf{f} \Box \iota_{\Delta})$. To make the notation simple in the following proof, let us write

$$\mathbf{g} = \mathrm{cl}(\mathbf{f} \Box \iota_{\Delta}).$$

Theorem 16

Let $\gamma \in]0,1[$, $\delta = 2-\gamma$, *and let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,\delta[$ *such that* $\sum_{n\in\mathbb{N}} \lambda_n(\delta-\lambda_n) = +\infty$. *Let* $\mathbf{x}_0 \in \mathbf{X}$ *and set*

$$\begin{array}{l} \textit{for } n = 0,1,\ldots \\ \left\lfloor \begin{array}{l} \mathbf{y}_n = (1-\gamma)\mathbf{x}_n + \gamma\mathbf{R}\mathbf{x}_n, \\ \mathbf{x}_{n+1} = \mathbf{x}_n + \lambda_n(\operatorname{prox}_{\gamma\mathbf{g}} \mathbf{y}_n - \mathbf{x}_n). \end{array} \right. \end{array} \tag{29}$$

Then the following hold:

1. $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *converges weakly to* $\mathbf{x}$, *a generalized cycle of* $\mathbf{g}$, *i.e., a solution of (EP).*
2. $(\mathbf{R}x_n - \mathbf{x}_n)_{n\in\mathbb{N}}$ *converges strongly to* $\mathbf{y} = \mathbf{R}\mathbf{x} - \mathbf{x}$, *the unique generalized gap vector of* $\mathbf{g}$, *i.e., the solution of (ED).*

Outline

In 2021, Simons considered the root of identity operator: $R : X \to X$ is linear and $R^m = \mathrm{Id}$.

Define the average operator

$$A = \frac{1}{m}\sum_{i=1}^{m} R^i, \text{ and } Y = \ker(A) = \{y \in X \mid Ay = 0\}.$$

Also define $S : X \to X$ by

$$S = R - \mathrm{Id}$$

and $Q : X \to X$ by

$$Q = \frac{1}{m}\sum_{i=1}^{m-1} iR^i,$$

and $Q_0 = Q|_Y$, the restriction of Q to Y. Linear operators A, S, Q and subspace Y are crucial in Simons' analysis [10].

Fact 17 (Simons '2021)

The following hold:

1. $S(X) \subseteq Y$, *and* $Q(Y) \subseteq Y$.
2. $(\forall y \in Y)\ S(Qy) = y,$ *and* $Q(Sy) = y$.
3. $AS = SA = 0$.
4. $SQ = QS = \operatorname{Id} - A$.
5. $-Q_0 - \operatorname{Id}/2$ *is skew and so maximally monotone on* Y.
6. *If* R *is an isometry, then* $(\forall x \in X)\ 2\langle x, Sx\rangle + \|Sx\|^2 = 0$.

Example 18

A linear operator $R : X \to X$ satisfying $R^m = \mathrm{Id}$ does not imply R nonexpansive. Let e_1, e_2, e_3, e_4 be the canonical base of the Euclidean space $\mathbb{R}^4$.

1. Bambaii–Chowla's matrix (1946): Set

$$B_1 = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   Then $B_1^5 = \mathrm{Id}$ but $\|B_1 e_1\| = \sqrt{2} > 1 = \|e_1\|$.

2. Set

$$B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

   Then $B_2^2 = \mathrm{Id}$ but $\|B_2 e_4\| = \sqrt{20} > 1 = \|e_4\|$.

3. Turnbull's matrix (1927): Set

$$B_3 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -3 & 2 & -1 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $B_3^3 = \mathrm{Id}$ but $\|B_3 e_1\| = \sqrt{20} > 1 = \|e_1\|$.

However, the following holds.

Lemma 19

Let $R : X \to X$ be linear and $R^m = \mathrm{Id}$ for $m \in \mathbb{N}$. Then the following are equivalent:

1. *R is nonexpansive.*
2. *R is an isometry.*
3. *R^* is nonexpansive.*
4. *R^* is an isometry.*

With Example 18 and Proposition 19 in mind, when R is an isometry we have the following new properties of A and S. We show that A is in fact a projection, and that

$$Y = (\operatorname{Fix} R)^{\perp} = \operatorname{ran} S$$

whenever R is an isometry.

Theorem 20

Suppose that R is an isometry. Then the following hold:

1. $\ker A = \ker A^* = (\operatorname{Fix} R)^{\perp} = (\operatorname{Fix} R^*)^{\perp}$.
2. $A = P_{\operatorname{Fix} R} = P_{\operatorname{Fix} R^*} = A^*$. *In particular,* $\operatorname{ran} A = \operatorname{ran} A^* = \operatorname{Fix} R$ *is closed.*
3. $\operatorname{ran} S = (\operatorname{Fix} R)^{\perp} = \operatorname{ran} S^*$. *In particular,* $\operatorname{ran} S = \operatorname{ran} S^*$ *is closed.*

Example 21

Without R being isometric, Theorem 20 fails. Take B_2 in Example 18(ii) where $m = 2$ to obtain

$$A = \frac{1}{2}(B_2 + B_2^2) = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & -1 & -3/2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Because

$$\|Ae_4\| = \sqrt{19/4} > \|e_4\|,$$

the operator A can neither be nonexpansive nor a projection operator.

Extended Simons' lemma

We call the following result the *extended Simons's lemma*.

Lemma 22

Let $f \in \Gamma_0(X)$ *with* $Y \cap \operatorname{dom} f^* \neq \varnothing$. *Then there exists a unique pair of vectors* $(e, d) = (e_f, d_f) \in Y \times Y$ *such that* $d = Se \in \operatorname{dom} f^*$, $e = Qd$, *and*

$$(\forall y \in Y)\ f^*(Se) + \langle y - Se, e\rangle - f^*(y) \leq 0;$$

equivalently, $e \in \partial(f^* + \iota_Y)(Se)$. *Consequently,*

$$(\forall x \in X)\ f^*(Se) + \langle Sx - Se, e\rangle - f^*(Sx) \leq 0.$$

• In [10, Lemma 16] Simons proved Lemma 22 when $f = \sigma_C$, a support function of a closed convex set $C \subseteq X$.

Lemma 23

Let $f \in \Gamma_0(X)$ *with* $Y \cap \operatorname{dom} f^* \neq \varnothing$. *Then the vector* $e = e_f \in Y$ *from Lemma 22 is the unique vector satisfying*

$$
\begin{aligned}
&(f^* + \iota_Y)(Se) - \langle Se, e\rangle + \mathrm{cl}(f \Box \iota_{Y^\perp})(e) &(30)\\
&= (f^* + \iota_Y)(Se) - \langle Se, e\rangle + (f^* + \iota_Y)^*(e) = 0. &(31)
\end{aligned}
$$

Theorem 24

Let R be an isometry and $Y = (\operatorname{Fix} R)^{\perp}$, let $f \in \Gamma_0(X)$ with $Y \cap \operatorname{dom} f^* \neq \varnothing$, and let $(e, d) \in Y \times Y$ be given by Lemma 22. Consider the Attouch–Théra primal-dual inclusion problem:

$$(P) \quad 0 \in \partial \operatorname{cl}(f \Box \iota_{Y^{\perp}})(x) + (\operatorname{Id} - R)x, \tag{32}$$

$$(D) \quad 0 \in \partial(f^* + \iota_Y)(y) + (\operatorname{Id} - R)^{-1} y. \tag{33}$$

Then the following hold:

1. (e, d) *is a solution to the primal-dual problem* (32)–(33), *i.e., e solves (P) and d solves (D). Moreover, d is the unique solution of (D).*
2. (e, d) *is the unique solution of the primal-dual problem*

$$(P') \quad 0 \in \partial \operatorname{cl}(f \Box \iota_{Y^{\perp}})(x) + (\operatorname{Id} - R)x \ \text{ and } \ x \in Y, \tag{34}$$

$$(D') \quad 0 \in \partial(f^* + \iota_Y)(y) + (\operatorname{Id} - R)^{-1} y. \tag{35}$$

More specifically, e is the unique solution of (P') and d is the unique solution of (D').

Outline

1. What is a cycle for a composition of proximal mappings?
2. Classical cylcles and gap vectors via the Attouch-Théra duality
3. Imagination: phantom cycle and gap vectors
4. Examples
5. Simons: *m*th roots of identity operator and an average operator
6. **Characterizations of classical cycle and gap vectors**
7. Phantom cycle and gap vectors for arbitrary isometry R
8. Conclusions

Theorem 25

Let $f \in \Gamma_0(X)$ with $Y \cap \operatorname{dom} f^* \neq \varnothing$ and let $(e, d) \in Y \times Y$ be given by Lemma 22. Then the following statements are equivalent for every $z \in X$:

1. $z = \operatorname{prox}_f Rz$.
2. $f^*(Sz) + f(z) + \frac{1}{2}\|Sz\|^2 = 0$.
3. $Sz = d$ *and* $f(z) = \operatorname{cl}(f \Box \iota_{Y^\perp})(e)$.
4. $Sz = d$ *and* $f(z) = \operatorname{cl}(f \Box \iota_{Y^\perp})(z)$.

Proof of Theorem 25 I

①⇔②: $z = \operatorname{prox}_f Rz \Leftrightarrow Rz \in z + \partial f(z) \Leftrightarrow Sz \in \partial f(z) \Leftrightarrow$

$$f^*(Sz) + f(z) = \langle z, Sz\rangle = -\frac{1}{2}\|Sz\|^2.$$

②⇒③: By ②,

$$f^*(Sz) + f(z) + \frac{1}{2}\|Sz\|^2 = 0. \tag{36}$$

By Lemma 22,

$$f^*(Se) + \langle Sz - Se, e\rangle - f^*(Sz) \leq 0.$$

Adding above two equations yields

$$f^*(Se) + f(z) + \langle Sz - Se, e\rangle + \frac{1}{2}\|Sz\|^2 \leq 0.$$

Since

$$f^*(Se) + f(z) \geq \langle Se, z\rangle,$$

Proof of Theorem 25 II

by the Fenchel–Young inequality, and

$$\frac{1}{2}\|Sz\|^2 = -\langle Sz, z\rangle,$$

we have

$$\langle Se, z\rangle + \langle Sz - Se, e\rangle - \langle Sz, z\rangle \leq 0,$$

from which

$$-\langle S(z-e), z-e\rangle = -\langle Sz - Se, z-e\rangle \leq 0.$$

Then $\frac{1}{2}\|S(z-e)\|^2 \leq 0$, so $Sz = Se = d$. Also, by Lemma 23 and $\langle Se, e\rangle = -\frac{1}{2}\|Se\|^2 = -\frac{1}{2}\|Sz\|^2$, we obtain

$$f^*(Sz) + \frac{1}{2}\|Sz\|^2 + \mathrm{cl}(f\Box\iota_{Y^\perp})(e) = 0. \tag{37}$$

Combining (36) and (37) gives $f(z) = \mathrm{cl}(f\Box\iota_{Y^\perp})(e)$.
③⇒②:

Proof of Theorem 25 III

Now ③ ensures $Sz = d = Se$ and $\mathrm{cl}(f \square \iota_{Y^\perp})(e) = f(z)$. Also

$$\langle Se, e\rangle = -\frac{1}{2}\|Se\|^2 = -\frac{1}{2}\|Sz\|^2.$$

Then (30) in Lemma 23 gives

$$f^*(Sz) + \frac{1}{2}\|Sz\|^2 + f(z) = 0,$$

which is ②.
③⇔④: Assume that $Sz = d = Se$. Then $z - e \in S^{-1}(0) = \operatorname{Fix} R$. Since $\mathrm{cl}(f \square \iota_{Y^\perp})$ is translation-invariant with respect to $Y^\perp = \operatorname{Fix} R$, we have

$$\mathrm{cl}(f \square \iota_{Y^\perp})(z) = \mathrm{cl}(f \square \iota_{Y^\perp})(e).$$

When does

$$f(z) = \mathrm{cl}(f \Box \iota_{Y^\perp})(e)$$

or

$$f(z) = \mathrm{cl}(f \Box \iota_{Y^\perp})(z)?$$

Translation-invariant functions

Definition 26

We say that $f : X \to]-\infty, +\infty]$ is translation-invariant with respect to a subset C of X if $f(x + c) = f(x)$ for every $x \in X$ and $c \in C$.

Lemma 27

Let $f \in \Gamma_0(X)$ and let C be a closed linear subspace of X. If f is translation-invariant with respect to C, then $\operatorname{dom} f^* \subseteq C^\perp$ and

$$(f^* + \iota_{C^\perp})^* = \operatorname{cl}(f \Box \iota_C) = f \Box \iota_C = f.$$

Theorem 28

Let $f \in \Gamma_0(X)$ be translation-invariant with respect to $\operatorname{Fix} R$ and such that $Y \cap \operatorname{dom} f^* \neq \varnothing$ where $Y = (\operatorname{Fix} R)^\perp$. Let $d \in Y$ be given by Lemma 22. Then the following statements are equivalent for every $z \in X$:

1. $z = \operatorname{prox}_f Rz$.
2. $f^*(Sz) + f(z) + \frac{1}{2}\|Sz\|^2 = 0$.
3. $Sz = d$.

Minimizers of f

Lemma 29

Let $f \in \Gamma_0(X)$ with $Y \cap \operatorname{dom} f^* \neq \varnothing$ and let $(e, d) \in Y \times Y$ be given by Lemma 22. Suppose in addition that $Sz = d$ and $z \in \operatorname{argmin} f$. Then

$$z = \operatorname{prox}_f Rz, \text{ and} \tag{38}$$

$$\operatorname{cl}(f \Box \iota_{Y^\perp})(e) = \operatorname{cl}(f \Box \iota_{Y^\perp})(z) = \min \operatorname{cl}(f \Box \iota_{Y^\perp}) = f(z). \tag{39}$$

Theorem 30

Let $f \in \Gamma_0(X)$ with $Y \cap \operatorname{dom} f^* \neq \varnothing$ and let $d \in Y$ be given by Lemma 22. Then the following statements are equivalent for every $z \in \operatorname{argmin} f$:

1. $z = \operatorname{prox}_f Rz$.
2. $f^*(Sz) + \frac{1}{2}\|Sz\|^2 + f(z) = 0$.
3. $Sz = d$.

Immediately we obtain the following result of Simons [10, Theorem 7].

Corollary 31

Let C be a nonempty closed convex subset of X. Let $d \in Y$ be given by Lemma 22 with $f = \iota_C$. Then the following statements are equivalent for every $z \in C$:

1. $z = P_C Rz$.
2. $\sigma_C(Sz) + \frac{1}{2}\|Sz\|^2 = 0$.
3. $Sz = d$.

Outline

The next result makes it clear that the classical cycles and gap vector of a function f are closely related to those of $\mathrm{cl}(f\Box\iota_{Y^\perp})$ and to which we refer as *phantom cycles* and *phantom gap vector*.

Theorem 32

Let $f \in \Gamma_0(X)$ with $Y \cap \operatorname{dom} f^* \neq \varnothing$ and let $(e, d) = (e_f, d_f)$ be given by Lemma 22. Then the following hold:

1. *The set Z of phantom cycles of f, which are defined to be the set of classical cycles of the function $\mathrm{cl}(f\Box\iota_{Y^\perp})$, i.e., $Z = \left\{z \in X \mid z = \mathrm{prox}_{\mathrm{cl}(f\Box\iota_{Y^\perp})}(Rz)\right\}$, is always nonempty and $Z = e + Y^\perp$. Consequently, Z contains infinitely many elements whenever $Y^\perp = \operatorname{Fix} R \neq \{0\}$.*
2. *The phantom gap vector of f, i.e., the gap vector $d_{\mathrm{cl}(f\Box\iota_{Y^\perp})}$, is equal to $d = Sz \in Y$ for every $z \in Z$; moreover, $e_{\mathrm{cl}(f\Box\iota_{Y^\perp})} = e$.*

Outline

Conclusions

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Conclusions

1. The Attouch-Théra duality provide a unified framework for studying cycles and gap vectors;
2. To define phantom cycles and gap vectors, one has to use $\mathrm{cl}(\mathbf{f}\Box\iota_{\Delta})$;
3. The forward-backward algorithms can be used to compute the phantom cycles and gap vectors;
4. How do we approach
$$0 \in \mathbf{Ax} + \mathbf{x} - \mathbf{Rx}$$
for a general maximally monotone operator $\mathbf{A}$?

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References III

Thank you!