Bargaining Schemes for the Split Cooperative Games

Shipra Singh

Department of Mathematical Sciences,
Norwegian University of Science and Technology,
Trondheim, Norway

April 4, 2022

Agenda

Core of Games

Bargaining Scheme

Some Fundamentals from Game Theory

- A game is a tool to model any situation in which players interact and take decision to attain a certain goal. In the game theory, there are two different approaches.
- One is the cooperative game approach started by Neumann and Morgenstern (1944), and the other is the non-cooperative game approach initiated by Nash (1950, 1953).
- In non-cooperative game approach, payers are supposed to choose their actions individually, and selfishly seek for their own goals and maximize their own profits.
- Cooperative game approach involves the players' alliances and their willingness to share their benefits obtained from their cooperation.

Any way, the question throughout this talk is not, “why to cooperate?", in fact it is, “how to cooperate?".

Core of a Cooperative Game

- $\Gamma_1(N, v)$; an $n-$person $(n > 2)$ cooperative game, and the game allows the agreements between two or more players,
- $N = \{1, 2, \ldots, n\}$; the set of n players,
- $P(N)$; set of all subsets of the player set N,
- S; non-empty subset of the set N is a possible alliance of the players,
- $v : P(N) \to \mathbb{R}$; the characteristic function of the game $\Gamma_1(N, v)$, which satisfies the following properties

$$v(\emptyset) = 0, \tag{1}$$

$$v(N) \geq \sum_{i=1}^{n} v(\{i\}). \tag{2}$$

- $v(S)$; worth of the alliance S. In other words, the alliance S can get maximum payoff (for simplicity, money) $v(S)$ without corresponding their strategies with the other $N \setminus S$ players,
- $v(N)$; maximum expected payoffs of the grand alliance N of the players,
- $\mathbb{R}^n$; $n-$dimensional Euclidean space with coordinates indexed by the elements of $N = \{1, 2, \ldots, n\}$,
- $x = (x_1, x_2, \ldots, x_n) = (x_i)_{i \in N} \in \mathbb{R}^n$; payoff vector of players. More precisely, the payoff vector $x = (x_1, x_2, \ldots, x_n)$ stands for the distribution of utilities available to the set of players in N is such that each player $i \in N$ receives the amount x_i. At some places, we also call the payoff vector x as imputation.

These distribution of utilities cannot be arbitrary, they must be governed by following certain restrictions:

$$x_i \geq v(\{i\}),\ \forall i = 1, 2, \dots, n, \tag{3}$$

$$\sum_{i=1}^{n} x_i = v(N). \tag{4}$$

Relation (3) is known as the condition of "individual rationality" because if this condition does not hold, then player i will definitely refuse to accept the distribution of payoffs since he or she is guaranteed to receive the amount $v(\{i\})$ without forming any alliances. Further, relation (4) is the condition of "group rationality". By rational behavior, we mean that players know what is the best way for them to obtain their goal in the game.

Indeed, the core is a solution concept used in cooperative game theory. By a solution we mean, how will the payoffs be distributed among the players in order to attain the "equilibrated payoff". We have already discussed both group and individual rationality. Further, imposing the collective rationality restrictions on the all possible alliances of players generates the solution concept known as the core. Let us assume that the core of game $\Gamma_1(N, v)$ is represented by $C(\Gamma_1)$, then mathematically it is expressed as following

$$C(\Gamma_1) = \{x \in I(v) : \sum_{i \in S} x_i \geq v(S),\ \forall S \subset N\}, \tag{5}$$

where $I(v) = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N)\}$.

Split Inverse Problem

A split inverse problem comprises a model in which two vector spaces X and Y, and a bounded linear operator $T : X \to Y$ are given. In addition, two inverse problems are involved. The first one, represented by IP_1, is formulated in the space X and the second one, represented by IP_2, is formulated in the space Y. Given these data, the split inverse problem is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1 \tag{6}$$

$$\text{and such that}$$

$$\text{the point } y^* = Ax^* \in Y \text{ solves } \mathrm{IP}_2. \tag{7}$$

Core of a Split Cooperative Game

First, we consider $\Gamma_2(M,u)$ is $m-$person cooperative game, $u : P(M) \to \mathbb{R}$ is its characteristic function and $C(\Gamma_2)$ is the core which is mathematically expressed as follows

$$C(\Gamma_2) = \{y \in I(u) : \sum_{j \in D} y_j \geq u(D),\ \forall D \subset M\}, \tag{8}$$

where $I(u) = \{y \in \mathbb{R}^m : \sum_{j=1}^{m} y_j = u(M)\}$.

Now, we define a split cooperative game which comprises two analogous cooperative games, $\Gamma_1(N,v)$ and $\Gamma_2(M,u)$. We hypothesize that the players of the game $\Gamma_2(M,u)$ determines the distribution of payoffs which comes from the linear transformation of the chosen distribution of payoffs by the players of the game $\Gamma_1(N,v)$. In continuation, we assume a bounded linear operator $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ and $AI(v) \subset I(u)$ which allows us to say that the games $\Gamma_1(N,v)$ and $\Gamma_2(M,u)$ are related.

We consider Γ represents the split cooperative game and $C(\Gamma)$ stands for its core, which is defined as follows

$$C(\Gamma) = \{x \in C(\Gamma_1) : Ax \in C(\Gamma_2)\}. \tag{9}$$

Special Case: If the number of players of the games $\Gamma_1(N, v)$ and $\Gamma_2(M, u)$ are same, i.e., $n = m$, then these games are called repetitive. In essence, the games are played over and over again for achieving equilibrated payoff vectors, and allotment of payoffs are newly chosen by the players, which are determined by the linear transformation of the allotments of the previous round of game.

2. Bargaining Scheme

Usually, we have two fundamental questions related to the concept of core in game theory: (1) how to determine whether the core of a given game is or is not empty?, (2) how to "better" redistribute the individual payoffs provided by an actual payoff vector, in order to reach an element of core when the core of the game is known to be nonempty? In this work, we derived all theories and results by keeping in the mind that core of game is nonempty. The reason behind considering the nonemptiness of core is, the knowledge that core of game is nonempty may provides the players with enough motivation "to bargain" for a "equilibrated" payoff vector in core by redistributing among themselves their actual payoffs.

We present a way of generating dynamic bargaining schemes which allow the players of a split cooperative game with nonempty core to redistribute their actual payoffs such that at the end of bargaining process an element of $C(\Gamma)$ is achieved no matter what the initial payoff was. The concept of bargaining scheme is essentially based on the idea that a bargaining scheme has to be a trajectory of a dynamic system.

A (set-valued) dynamic system on $\mathbb{R}^n$ is a set-valued function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ which satisfies $\phi(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$. A bargaining scheme, or a bargaining process, or a dynamic process, or a trajectory, starting at $x^\circ \in \mathbb{R}^n$ is a sequence $x^\circ, x^1, \ldots$ having the property that $x^{k+1} \in \phi(x^k),\ k = 0, 1, \ldots$.
we can explain as, in the classical physical science, usually a (dynamic) process is determined by the initial conditions. If x° is the "state of world" at time $t = 0$ (initial position) and x^1 is the "state of world" at time $t = 1$. Now, if a decision making person has more than one option in a dynamic system ϕ, one cannot, in general, predict the "state of the world" at time $t = 1$ from the only knowledge of the "state of the world" at time $t = 0$. All one can say is that the state at $t = 1$ will belong to a set of states $\phi(x^\circ)$, i.e., $x^1 \in \phi(x^\circ)$.

- $e(S,x) = v(S) - \sum_{i \in S} x_i$; excess of a nonempty alliance S with respect to the imputation $x \in \mathbb{R}^n$ in the game $\Gamma_1(N,v)$,
- $f(D,y) = u(D) - \sum_{j \in D} y_j$; excess of a nonempty alliance D with respect to the imputation $y \in \mathbb{R}^m$ in the game $\Gamma_2(M,u)$,
- for any $x \in \mathbb{R}^n$ denote by $\Pi_1(x)$ the subset of $P(N)$ of all non-empty alliances S from the game $\Gamma_1(N,v)$, which satisfies the following

$$\frac{e(S,x)}{|S|} = \max\left\{\frac{e(V,x)}{|V|} : \emptyset \neq V \subseteq N\right\}, \tag{10}$$

- for any $y \in \mathbb{R}^m$ denote by $\Pi_2(y)$ the subset of $P(M)$ of all non-empty alliances D from the game $\Gamma_2(M,u)$, which satisfies the following

$$\frac{f(D,y)}{|D|} = \max\left\{\frac{f(U,y)}{|U|} : \emptyset \neq U \subseteq M\right\}. \tag{11}$$

- $\chi_k = \{x^k : k = 0, 1, \ldots\}$; a bargaining scheme with respect to a (set-valued) dynamic system ϕ which satisfies $x^{k+1} \in \phi(x^k),\ k = 0, 1, \ldots$.,
- A point $x \in \mathbb{R}^n$ is called an end point of the dynamic system ϕ if $\phi(x) = \{x\}$. Further, we say that the dynamic system ϕ is a bargaining system for the subset Q of $\mathbb{R}^n$ if the bargaining scheme χ_k is convergent and their limits are contained in Q.

Further, we define a set-valued function ϕ^1 over $\mathbb{R}^n$ by

$$\phi^1(x) = \begin{cases} \text{proj}_{X_S}(x + \gamma A^T(\text{proj}_{Y_D}(Ax) - Ax)), \ \forall S \in \Pi_1(x) \text{ and} \\ \qquad D \in \Pi_2(Ax), \text{ if } e(N,x) \geq 0 \text{ and } f(M,Ax) \geq 0, \\ \text{proj}_{I(v)}(x + \gamma A^T(\text{proj}_{I(u)}(Ax) - Ax)), \text{ if } e(N,x) < 0 \text{ and} \\ \qquad f(M,Ax) < 0, \end{cases}$$

where A^T is the transpose of A, $0 < \gamma < L_1$ and L_1 is the largest eigenvalue of the matrix A^TA, $X_S = \{x \in \mathbb{R}^n : e(S,x) \leq 0\}$ for the alliance $S \subset N$ and $Y_D = \{y \in \mathbb{R}^m : f(D,y) \leq 0\}$ for the alliance $D \subset M$.

Theorem

If $C(\Gamma) \neq \emptyset$ then the the (set-valued) dynamic system ϕ^1 is a bargaining system for $C(\Gamma)$, and the points of $C(\Gamma)$ are the end points of the bargaining system ϕ^1.

Let's say χ_k^1 is the bargaining scheme with respect to the (set-valued) dynamic system ϕ^1. Now, the bargaining scheme χ_k^1 works in the sense that it is a model of bargaining procedures in which at each stage $k = 0, 1, 2, \ldots$ of the bargaining process, a single alliance (usually one of the better performing in previous stages) redistributes its whole excess to its member, and this implicitly means that there exists a pre-negotiated agreement (defined as sequential rule $x^{k+1} \in \phi(x^k)$) of how to select the alliance at each bargaining stage which will pay for improving the actual payoff vector in order to obtain equilibrated payoff vector, i.e., the element of core $C(\Gamma)$.

The bargaining scheme χ_k^1, no matter how the initial payoff x° is chosen, is based upon the following principle: the players agree that if at a stage k of the bargaining process, when the current actual payoff vectors are x^k and Ax^k, and their total alliances N and M has a loss (i.e., $e(N,x^k) < 0$ and $f(M,Ax^k) < 0$) in the game $\Gamma^1(N,v)$ and $\Gamma^2(M,u)$, respectively, then these losses have to be reimbursed, that is the losses have to be redistributed equally among all the players in their respective games (each player has to pay the amount $\frac{-e(N,x^k)}{n}$ and $\frac{-f(M,Ax^k)}{m}$ in their respective games). Otherwise, one of the best performing alliances (that is one of the alliances having minimal marginal loss or equivalently, maximal marginal excess, see 10 and 11) at the specific stage of the bargaining process redistributes its excess equally among its members (that is each member of one of the best performing coalition S and D is paid the amount $\frac{e(S,x^k)}{|S|}$) and $\frac{f(D,Ax^k)}{|D|}$ in their respective games).

In this way, after stage k, the actual payoff vector becomes

$$x^{k+1} = \begin{cases} \text{proj}_{X_S}(x^k + \gamma A^T(\text{proj}_{Y_D}(Ax^k) - Ax^k)), \text{ if } e(N,x^k) \geq 0 \text{ and} \\ \qquad f(M,Ax^k) \geq 0, \\ \text{proj}_{I(v)}(x^k + \gamma A^T(\text{proj}_{I(u)}(Ax^k) - Ax^k)), \text{ if } e(N,x^k) < 0 \text{ and} \\ \qquad f(M,Ax^k) < 0, \end{cases}$$

where S and D are one of the best performing alliance of the players at the bargaining stage k.

THANK YOU!