A Neural Network for Solving Inverse Quasi-Variational Inequalities

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April 4, 2022

Variational inequality problem

- Suppose we want to minimize a differentiable function, say $\varphi : [a, b] \to \mathbb{R}$, where $[a, b] \subseteq \mathbb{R}$. If x_0 is a solution, then a necessary condition is the following:
  - (i) $\varphi'(x_0) = 0$ if $a < x_0 < b$.
  - (ii) $\varphi'(x_0) \geq 0$ if $x_0 = a$.
  - (iii) $\varphi'(x_0) \leq 0$ if $x_0 = b$.

  The above can be combined as

$$\langle \varphi'(x_0), y - x_0 \rangle \geq 0, \forall y \in [a, b].$$

- Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty closed convex subset and $f : \Omega \to \mathbb{R}^n$. Then the *variational inequality problem* $VI(f, \Omega)$ is to find an $x^* \in \Omega$ such that

$$\langle f(x^*), y - x^* \rangle \geq 0 \quad \forall y \in \Omega. \tag{1}$$

[1] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C.R. Acad. Sci. Paris, 258 (1964), 4413-4416.

- It is easy to see that the VI (1) is equivalent to the following projection equation

$$x^* = P_\Omega(x^* - \alpha f(x^*)) = P_\Omega(I - \alpha f)x^*, \tag{2}$$

where $\alpha > 0$ is a constant and P_Ω is the metric projection onto Ω defined by

$$P_\Omega(x) = \arg\min_{y \in \Omega} \|x - y\|^2 \quad \forall x \in \mathbb{R}^n.$$

Theorem 1.1

Let Ω be a non-empty compact and convex subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be a continuous function. Then there exists a solution to the variational inequality (1).

Variational Inequality Problem and its generalizations

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Inverse variational inequality problem

An important generalization is known as inverse variational inequality problem.

- Let $\mathbb{R}^n$ denote the real n-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. An *inverse variational inequality* (IVI) is to find an $x^* \in \mathbb{R}^n$ such that

$$F(x^*) \in \Omega \quad \text{and} \quad \langle x^*, y - F(x^*) \rangle \geq 0 \quad \forall y \in \Omega, \tag{3}$$

  where F is a mapping from $\mathbb{R}^n$ into itself and Ω is a nonempty closed convex subset of $\mathbb{R}^n$.

- If an inverse function $x = F^{-1}(u) = f(u)$ exists, then the above IVI problem can be transformed into the following regular variational inequality: find a point $u^* \in \Omega$ such that

$$\langle f(u^*), v - u^* \rangle \geq 0 \quad \forall v \in \Omega. \tag{4}$$

[2] B. He, X. He, H. X. Liu, Solving a class of constrained 'black-box' inverse variational inequalities, Eur. J. Oper. Res., 204(3) (2010) 391-401.

- Similarly, it is easy to see that the IVI (3) is equivalent to the following projection equation

$$F(x^*) = P_\Omega(F(x^*) - \alpha x^*), \tag{5}$$

where $\alpha > 0$ is a constant and P_Ω is the metric projection onto Ω defined by

$$P_\Omega(x) = \arg\min_{y \in \Omega} \|x - y\|^2 \quad \forall x \in \mathbb{R}^n.$$

- Zou, et. al. [3] proposed a neural network method to solve IVI (3) by considering the following neural network:

$$\dot{x} = \lambda \{P_\Omega(F(x) - \alpha x) - F(x)\} =: G(x), \tag{6}$$

where $\dot{x} = \frac{dx}{dt}$ and $\lambda > 0$ is a fixed parameter.

[3] X. Zou, D. Gong, L. Wang, Z. Chen, A novel method to solve inverse variational inequality problems based on neural networks, Neurocomputing, 173 (2016), 1163-1168.

Definition 2.1

A dynamical system is said to be *globally exponentially* stable at x^* if every trajectory starting at any initial point $x(t_0) \in \mathbb{R}^n$ satisfy

$$\|x(t) - x^*\| \leq \gamma e^{-\zeta(t-t_0)}, \quad \forall t \geq t_0,$$

where γ and ζ are positive constants independent of the initial point.

Definition 2.2

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be β*-strongly monotone* if for some $\beta > 0$,

$$\langle f(x) - f(y), x - y \rangle \geq \beta \|x - y\|^2 \qquad \forall x, y \in \mathbb{R}^n.$$

[4] H. H. Bauschke, P. L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, New York, Dordrecht, Heidelberg, London, 2011.

Theorem 2.3

Consider IVI (3) with its nonempty solution set Ω^* and its associated neural network (6), where we assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous mapping. Assume, in addition that F is a symmetric gradient mapping, that is, $F = \nabla g$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex, continuously differentiable function. Then the following conclusions hold.

(i) *Every trajectory $x(t)$ of the neural network (6) converges to the set of equilibrium points; that is,*

$$\lim_{t\to\infty} \operatorname{dist}(x(t), \Omega^*) = 0. \tag{7}$$

Equivalently, each cluster point of a trajectory $x(t)$ is an equilibrium point. [Here $\operatorname{dist}(x, \Omega^*) := \inf\{\|x - z\| : z \in \Omega^*\}$ is the distance from x to Ω^*.]

(ii) If IVI (3) has a unique solution x^*, i.e., $\Omega^* = \{x^*\}$, then the neural network (6) is globally asymptotically stable at the equilibrium point x^*.

(iii) If F is strongly monotone, then IVI (3) must have a unique solution x^* and further the neural network (6) is globally exponentially stable at the equilibrium point x^*.

[5] Hong-Kun Xu, Soumitra Dey, V Vetrivel, Notes On the Neural Network Approach to Inverse variational Inequalities, Optimization, 70(5-6) (2021), 901-910.

Inverse quasi-variational inequality problem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued mapping and $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping. The *inverse quasi-variational inequality problem* (IQVIP) is to find a vector $x \in \mathbb{R}^n$ such that

$$f(x) \in \Phi(x), \quad \langle x, y - f(x) \rangle \geq 0, \forall y \in \Phi(x). \tag{8}$$

- In the case where the set-valued mapping $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ which governs the IQVIP (8) has nonempty, closed and convex point values, it is not difficult to check that x^* is a solution to (8) if and only if it is a solution to the projection equation

$$f(x) = P_{\Phi(x)}(f(x) - \alpha x), \tag{9}$$

where $\alpha > 0$ is a fixed constant.

[4] D. Aussel, R. Gupta, and A. Mehra, *Gap functions and error bounds for inverse quasi-variational inequality problems*, J. Math. Anal. Appl., **407** (2013), 270-280 .

Neural netwoek for the IQVIP

- Motivated by the above neural network approaches to the IVI, here we introduce the following network to solve IQVIP:

$$\dot{x} = \lambda(t)\left\{P_{\Phi(x)}(f(x) - \alpha x) - f(x)\right\} = S(x,t), \tag{10}$$

  where $\dot{x} = \frac{dx}{dt}$ and $\lambda(t) > 0$, $t \geq 0$, are parameters.
- In particular, if $\lambda(t) = \lambda$ for every $t \in [0, \infty)$ and $\Phi(x) = \Omega$ for every $x \in \mathbb{R}^n$, then the network (10) reduces to the network (6), which was considered and studied in ([3, 4]).

Basic definitions and results

Lemma 2.4

Let Ω be a nonempty, closed and convex subset of $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $z \in \Omega$, we have

$$z = P_\Omega(x) \iff \langle x - z, y - z \rangle \leq 0 \quad \forall y \in \Omega. \tag{11}$$

It turns out that the projection operator P_Ω is nonexpansive.

Theorem 2.5

(Banach's fixed point theorem) Let X be a Banach space and let $f : X \to X$ be a strict contraction. Then f has a unique fixed point.

Our main results

Lemma 2.6

Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex point values. Then we have

$$\begin{aligned} &\|u - P_{\Phi(x)}(u) - (v - P_{\Phi(y)}(v))\| \\ &\leq \|u - v\| + \|P_{\Phi(x)}(v) - P_{\Phi(y)}(v)\| \quad \forall x, y, u, v \in \mathbb{R}^n. \end{aligned} \tag{12}$$

Sketch of the proof:

Let $x, y, u, v \in \mathbb{R}^n$. Using Lemma 2.4, we see that

$$\langle v - P_{\Phi(x)}(v), P_{\Phi(x)}(u) - P_{\Phi(x)}(v) \rangle \leq 0 \tag{13}$$

and

$$\langle u - P_{\Phi(x)}(u), P_{\Phi(x)}(v) - P_{\Phi(x)}(u) \rangle \leq 0. \tag{14}$$

continuation of the proof

Adding (13) and (14), we get

$$\|P_{\Phi(x)}(u) - P_{\Phi(x)}(v)\|^2 \leq \langle P_{\Phi(x)}(u) - P_{\Phi(x)}(v), u - v \rangle . \tag{15}$$

It follows from (15) that

$$\begin{aligned} &\|u - P_{\Phi(x)}(u) - (v - P_{\Phi(x)}(v))\|^2 \\ &\leq \|u - v\|^2 \end{aligned} \tag{16}$$

and so,

$$\begin{aligned} &\|u - P_{\Phi(x)}(u) - (v - P_{\Phi(y)}(v))\| \\ &\leq \|u - v\| + \|P_{\Phi(x)}(v) - P_{\Phi(y)}(v)\|. \end{aligned} \tag{17}$$

Theorem 2.7

Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex point values. Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is $L-$Lipschitz continuous and $\beta-$strongly monotone. Assume further that there exists some $\kappa > 0$ such that

$$\|P_{\Phi(x)}(z) - P_{\Phi(y)}(z)\| \leq \kappa\|x - y\| \quad \forall x, y, z \in \mathbb{R}^n \tag{18}$$

and $L^2 - 2\alpha(\beta - \kappa) < \kappa^2$, where $\alpha > 0$ is a constant. Then the inverse quasi-variational inequality problem (8) has a unique solution.

Sketch of the proof

Define a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$h(x) = x - \frac{1}{\alpha} f(x) + \frac{1}{\alpha} P_{\Phi(x)}(f(x) - \alpha x),$$

where $\alpha > 0$ is a fixed constant.
It is clear that x^* is a solution to the inverse quasi-variational inequality if and only if x^* is a fixed point of the mapping h.
Let $\bar{x} = f(x) - \alpha x$ and $\bar{y} = f(y) - \alpha y$. Using Lemma 2.6, we see that

$$\|h(x) - h(y)\| \leq \frac{1}{\alpha}(\|\bar{x} - \bar{y}\| + \kappa\|x - y\|). \tag{19}$$

Continuation of the proof

Now,

$$\begin{aligned} \|\bar{x}-\bar{y}\|^2 &= \|f(x)-\alpha x-(f(y)-\alpha y)\|^2 \\ &\leq (L^2-2\beta\alpha+\alpha^2)\|x-y\|^2. \end{aligned} \tag{20}$$

Using (19) and (20) we get,

$$\|h(x)-h(y)\| \leq \frac{1}{\alpha}\left(\sqrt{(L^2-2\beta\alpha+\alpha^2)}+\kappa\right)\|x-y\|. \tag{21}$$

It clearly follows from our assumptions that h is a strict contraction with constant $(\sqrt{(L^2-2\beta\alpha+\alpha^2)}+\kappa)/\alpha \in [0,1)$. Therefore, by the Banach contraction principle (Theorem 2.5), the mapping h has a unique fixed point. In other words, the inverse quasi-variational inequality problem (8) has a unique solution.

Remark 2.1

Note that assumption (18) is a kind of contraction property for the set-valued mapping Φ on $\mathbb{R}^n$. In several applications the point image can be written as

$$\Phi(x) = s(x) + \Omega,$$

where $s(x)$ is a Lipschitz continuous single-valued mapping from $\mathbb{R}^n$ into itself with Lipschitz constant λ and Ω is a closed convex subset of $\mathbb{R}^n$. In this case, the assumption (18) holds with the same Lipschitz constant value of λ.

Existence and uniqueness of a solution to (10)

Theorem 2.8

Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex point images and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous mapping with Lipschitz constant L. Assume that there exists a number $\kappa > 0$ such that

$$\|P_{\Phi(x)}(z) - P_{\Phi(y)}(z)\| \leq \kappa\|x - y\| \quad \forall x, y, z \in \mathbb{R}^n. \tag{22}$$

Then the dynamical system (10) a has unique solution.

Proof.

We claim that $S(x,t)$ is Lipschitz continuous for all fixed $t \geq 0$. Indeed, we have

$$
\begin{aligned}
&\|S(x,t) - S(y,t)\| \\
&= \|\lambda(t)(P_{\Phi(x)}(f(x) - \alpha x) - f(x)) - \lambda(t)(P_{\Phi(y)}(f(y) - \alpha y) - f(y))\| \\
&\leq (2L + \alpha + \kappa)\lambda(t)\|x - y\|.
\end{aligned}
$$

Furthermore, if $\lambda(t)$ is continuous, then the function $S(x,\cdot)$ is continuous for all fixed $x \in \mathbb{R}^n$ and the differential equation (10), for arbitrary initial points $x_0 \in \mathbb{R}^n$, has a unique solution for all $t \geq t_0 \geq 0$. □

[7] P. Hartman, Ordinary Differential Equations, Classics in Applied Mathematics, Vol. 18. SIAM, Philadelphia, 2002.

- The evolution function m^t is often the solution of a differential equation of motion:

$$\dot{x} = h(x). \tag{23}$$

Recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a Lyapunov function (about $x = x_e$) for the dynamical system (23) if the following three properties are satisfied:

(L1) V is positive definite, namely, $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(x) = 0$ if and only if $x = x_e$;

(L2) $\dot{V}$ is negative definite along the trajectories of (23), that is, if $x(t)$ is a trajectory of (23), then $\dot{V}(x(t)) \leq 0$ for all $t \geq 0$ and $\dot{V}(x(t)) < 0$ for all $x \neq x_e$;

(L3) V is coercive (also known as radially unbounded), that is, $V(x) \to \infty$ as $\|x\| \to \infty$.

Theorem 2.9

(Lyapunov's Theorem) Let x_e be an equilibrium of the dynamical system (23). If there exists a Lyapunov function about x_e, then the dynamical system (23) globally asymptotically stable at the equilibrium point x_e.

Stability analysis

Theorem 2.10

Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued map with nonempty, closed and convex point values and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous with Lipschitz constant L and β-strongly monotone. Assume that the parameters $\lambda(t) \in C([0, \infty))$. Assume that

$$1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta < 0,$$

$$\int_{t_0}^{\infty} \lambda(t)dt = +\infty$$

and

$$L^2 - 2\alpha(\beta - \kappa) < \kappa^2,$$

where κ satisfies

$$\|P_{\Phi(x)}(z) - P_{\Phi(y)}(z)\| \leq \kappa\|x - y\| \quad \forall x, y, z \in \mathbb{R}^n.$$

Continuation

Then the dynamical system (10) converges to the solution of IQVIP (8) at the rate

$$\|x(t) - x^*\| \leq \|x_0 - x^*\| \exp \int_{t_0}^{t} \Lambda(t) dt,$$

where $\Lambda(t) = \lambda(t)[1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta]$.
Furthermore, the dynamical system (10) is globally asymptotically at the point of equilibrium x^*. In addition, if $\lambda(t) \geq \lambda^* > 0$ for every $t \geq 0$, then the dynamical system (10) is globally exponentially stable at the point of equilibrium x^*.

Sketch of the proof

- Using Theorem 2.8, we can easily show that (10) has a unique solution. Let x^* be the unique solution of the IQVIP (8). Now we have to show that the trajectories of the network are globally asymptotically and exponentially stable at the equilibrium point x^*. To this end, consider the Lyapunov function

$$V(x) = \|x - x^*\|^2.$$

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- $$\begin{aligned} V'(x) &= 2\langle x - x^*, x' \rangle \\ &= 2\left\langle x - x^*, \lambda(t)(P_{\Phi(x)}(f(x) - \alpha x) - f(x)) \right\rangle \\ &\leq \lambda(t)[1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta]V(x) = \Lambda(t)V(x), \end{aligned}$$

  where $\Lambda(t) = \lambda(t)[1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta]$.
- Since $\int_{t_0}^{\infty} \lambda(t)dt = +\infty$ and $1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta < 0$, we see that $\int_{t_0}^{\infty} \Lambda(t)dt = -\infty$. Hence $\exp \int_{t_0}^{\infty} \Lambda(t)dt = 0$.
- Consequently, the trajectory $x(t)$ converges to the unique solution x^* of (8) and it is not difficult to show that

$$\|x(t) - x^*\| \leq \|x_0 - x^*\| \exp \int_{t_0}^{t} \Lambda(t)dt. \qquad (24)$$

- It now follows from Theorem 2.9 that the dynamical system (10) is globally asymptotically stable at the equilibrium point x^*.
- If $\lambda(t) \geq \lambda^* > 0$ for every $t \geq 0$, from (24) we get

$$\|x(t) - x^*\| \leq \|x_0 - x^*\| e^{-\zeta(t-t_0)} \quad \forall t \geq t_0,$$

  where $\zeta = -\lambda^*(1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta) > 0$. Therefore, the dynamical system (10) is globally exponentially stable at the equilibrium point x^*.

Discretization of the Network (10)

The explicit discretization of the neural network (10) with respect to t with the step-size h_n and with the initial point $x_0 \in \mathbb{R}^n$ is as follows:

$$\frac{x_{n+1} - x_n}{h_n} = \lambda_n \{P_{\Phi(x_n)}(f(x_n) - \alpha x_n) - f(x_n)\}, \tag{25}$$

If $h_n = 1$, then the above scheme reduces to the following one:

$$x_{n+1} = x_n + \lambda_n \{P_{\Phi(x_n)}(f(x_n) - \alpha x_n) - f(x_n)\}. \tag{26}$$

Theorem 2.11

Assume that

1. $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* β*-strongly monotone and L-Lipschitz continuous;*
2. $\Phi(x) = s(x) + \Omega$, *where* $s : \mathbb{R}^n \to \mathbb{R}^n$ *is a Lipschitz continuous mapping with Lipschitz constant* $l > 0$ *and* Ω *is a nonempty, closed and convex subset of* $\mathbb{R}^n$*;*
3. $$l < \beta, \frac{L^2 + l^2}{2(\beta - l)} < \alpha; \tag{27}$$
4. *for every* $n \in \mathbb{N}$,

$$0 < A < \lambda_n < B, \quad \text{where} \tag{28}$$

$$0 < \frac{B^2}{A} < \frac{(2\alpha(\beta - l) - (L^2 + l^2))}{\alpha^2(\beta - l)}. \tag{29}$$

Then the sequence $\{x_n\}$ *generated by (26) converges strongly to the unique solution of the IQVIP (8).*

Sketch of the proof:

- Using our above assumptions, and the existence and uniqueness Theorem 2.7 for the IQVIP (8), we conclude that it has a unique solution. Let x^* be the unique solution of the IQVIP (8). Recalling (26), we have

$$x_{n+1} = x_n + \lambda_n \{P_{\Phi(x_n)}(f(x_n) - \alpha x_n) - f(x_n)\}$$

-

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|x_n - x^* + \lambda_n \{P_{\Phi(x_n)}(f(x_n) - \alpha x_n) - f(x_n)\}\|^2 \qquad (30) \\ &= \|x_n - x^*\|^2 + \lambda_n^2 \|P_{\Phi(x_n)}(f(x_n) - \alpha x_n) - f(x_n)\|^2 \\ &+ 2\lambda_n \left\langle x_n - x^*, P_{\Phi(x_n)}(f(x_n) - \alpha x_n) - f(x_n) \right\rangle . \end{aligned}$$

- Setting $y_n = P_{\Phi(x_n)}(f(x_n) - \alpha x_n)$, we see that we need to approximate $\|y_n - f(x_n)\|^2$.
  Note that

$$\begin{aligned} y_n &= P_{s(x_n)+\Omega}(f(x_n) - \alpha x_n) \\ &= s(x_n) + P_{\Omega}(f(x_n) - \alpha x_n - s(x_n)), \end{aligned} \tag{31}$$

  Therefore, using the characterization of the nearest point projection, we see that, for any $n \in \mathbb{N}$,

$$\langle f(x_n) - \alpha x_n - y_n, z - y_n + s(x_n)\rangle \leq 0 \quad \forall z \in \Omega. \tag{32}$$

- Since x^* is a solution of the IQVIP (8), we have $x^* \in \mathbb{R}^n$ and

$$f(x^*) = P_{\Phi(x^*)}(f(x^*) - \alpha x^*). \tag{33}$$

- Using (33), we get

$$\langle \alpha x^*, f(x^*) - y_n + s(x_n) - s(x^*) \rangle \leq 0 \quad \forall n \in \mathbb{N}. \tag{34}$$

- Therefore it follows from (32) that

$$\langle f(x_n) - \alpha x_n - y_n, f(x^*) - y_n + s(x_n) - s(x^*) \rangle \leq 0 \quad \forall n \in \mathbb{N}. \tag{35}$$

- Combining (34) and (35), we have

$$\langle f(x_n) - \alpha x_n - y_n + \alpha x^*, f(x^*) - y_n + s(x_n) - s(x^*) \rangle \leq 0 \quad \forall n \in \mathbb{N} \tag{36}$$

- Simplifying (36), we get

$$\begin{aligned} \|f(x_n) - y_n\|^2 \leq \langle f(x_n) - y_n, f(x_n) - f(x^*) \rangle + \langle y_n - f(x_n), s(x_n) - s(x^*) \rangle \quad (37) \\ + \alpha \langle x_n - x^*, f(x_n) - y_n \rangle - \alpha \langle x_n - x^*, f(x_n) - f(x^*) \rangle \\ + \alpha \langle x_n - x^*, s(x_n) - s(x^*) \rangle . \end{aligned}$$

- Since f is L-Lipschitz continuous and β-strongly monotone, s is l-Lipschitz continuous. Using inequality (37), we get

$$\langle x_n - x^*, y_n - f(x_n)\rangle \leq \frac{1}{\alpha}(\alpha(l-\beta) + \frac{L^2+l^2}{2})\|x_n - x^*\|^2, \qquad (38)$$

- Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ in (37) and (27), we obtain

$$\alpha\langle x_n - x^*, y_n - f(x_n)\rangle \leq \frac{(L^2+l^2) - 2\alpha(\beta - l)}{2\alpha(\beta - l)}\|y_n - f(x_n)\|^2. \qquad (39)$$

- Using (38), (28) and (29) we obtain the following inequality:

$$\|x_{n+1} - x_n\|^2 \leq \|x_n - x^*\|^2 - \Big(2\lambda_n - \frac{2\lambda_n^2\alpha^2(\beta - l)}{2\alpha(\beta - l) - (L^2 + l^2)}\Big)\Big(\frac{2\alpha(\beta - l) - L^2 - l^2}{2\alpha}\Big)\|x_n - x^*\|^2. \tag{40}$$

- This implies that

$$\|x_{n+1} - x^*\| \leq \sqrt{1 - \Big(2\lambda_n - \frac{2\lambda_n^2\alpha^2(\beta - l)}{2\alpha(\beta - l) - (L^2 + l^2)}\Big)\Big(\frac{2\alpha(\beta - l) - L^2 - l^2}{2\alpha}\Big)}\|x_n - x^*\|. \tag{41}$$

- Define for every $n \in \mathbb{N}$,

$$Q(\alpha, \lambda_n) := \sqrt{1 - \left(2\lambda_n - \frac{2\lambda_n^2\alpha^2(\beta - l)}{2\alpha(\beta - l) - (L^2 + l^2)}\right)\left(\frac{2\alpha(\beta - l) - L^2 - l^2}{2\alpha}\right)}.$$

Finally, we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq Q(\alpha, \lambda_n)\|x_n - x^*\| \qquad (42) \\ &\vdots \\ &\leq Q^n(\alpha, \lambda_n)\|x_0 - x^*\|. \end{aligned}$$

- Let $C_1 = \frac{2\alpha(\beta-l)-(L^2+l^2)}{\alpha}$ and $C_2 = \alpha(\beta - l)$. Then we have

$$Q(\alpha, \lambda_n)^2 = 1 + \lambda_n^2 C_2 - \lambda_n C_1. \tag{43}$$

  Using facts (28) and (29), we get

$$Q(\alpha, \lambda_n)^2 < 1 + B^2 C_2 - AC_1 = r < 1.$$

  Again, from (42) we infer that

$$0 \leq \|x_{n+1} - x^*\| < r^{n/2}\|x_0 - x^*\| \quad \forall n \in \mathbb{N}. \tag{44}$$

The proof is complete.

Remark 2.2

In particular, if we take $\lambda_n = \frac{1}{\alpha}$ *for every* $n \in \mathbb{N}$ *and* $\Phi(x) = \Omega$ *(that is,* $s(x) = 0$*) for every* $x \in \mathbb{R}^n$*, then (26) reduces to the algorithm*

$$x_{n+1} = x_n + \frac{1}{\alpha}\{P_\Omega(f(x_n) - \alpha x_n) - f(x_n)\}, \tag{45}$$

which was studied by He et al. [4].
In this case, the conditions on parameters of Theorem 2.11 reduces to $\alpha > \frac{L^2}{\beta}$ *and the sequence* $\{x_n\}$ *satisfies the following inequality:*

$$\|x_{n+1} - x^*\| \leq \sqrt{\left(1 - \frac{\alpha\beta - L^2}{\alpha^2}\right)}\|x_n - x^*\|, \tag{46}$$

where β *and* L *are the strong monotonicity and Lipschitz constants, respectively, of the function* f*. Therefore, the iterative sequence generated in (46) converges to* x^**.*

[4] X. He, H. X. Liu, Inverse variational inequalities with projection-based solution methods, Eur. J. Oper. Res., 208 (2011) 12-18.

An example

- Let $\Omega = B[0,1] \subset \mathbb{R}^3$, the closed unit ball centered at the origin. Consider the functions $f(x) = 2x$ and $s(x) = x/4$ from $\mathbb{R}^3$ into itself. Let $\alpha = 2$ and $\lambda(t) = 1 + t^3$, where $t \geq 0$. Then we have $\lambda(t) \in C([0, +\infty))$ and

$$\int_{t_0=0}^{\infty} \lambda(t)dt = +\infty.$$

Let $\Phi(x) = s(x) + \Omega$.

- It can be verified that the above parameters satisfy the following conditions:

$$1 + 2\kappa - 2\beta + \alpha^2 + L^2 - 2\alpha\beta < 0 \quad \text{and}$$
$$L^2 - 2\alpha(\beta - \kappa) < \kappa^2.$$

- Using Theorem 2.7, it not difficult to check that $(0, 0, 0)$ is the unique equilibrium point for the neural network (10), that is, $(0, 0, 0)$ is the unique solution of the IQVIP (8). According to Theorem 2.10, the neural network is globally asymptotically and exponentially stable at $(0, 0, 0)$. The graph below shows that the trajectories of (10) globally converge to the optimal solution $(0, 0, 0)$ with different starting points. Furthermore, we see that the corresponding neural network converges at a faster rate.

Variational Inequality Problem and its generalizations

Soumitra Dey

Variational inequality problem

Inverse variational inequality problem

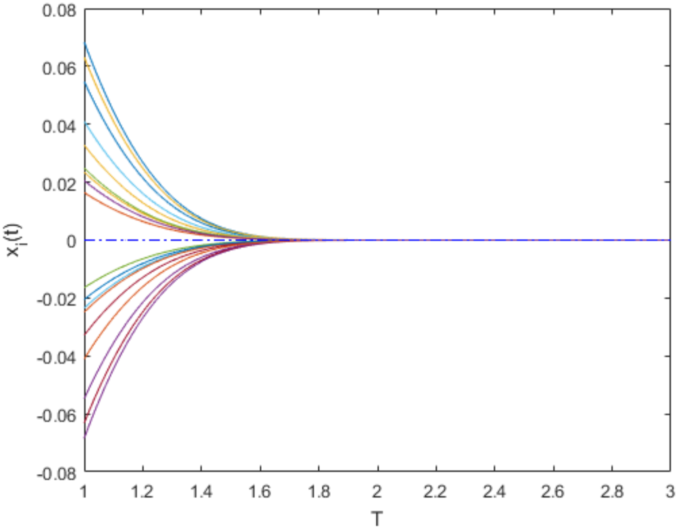

Fig. 1 Transient behavior of the neural network (10).

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Conclusions

- We have proved an existence and uniqueness theorem for the IQVIP (8).
- We presents a recurrent neural network model for solving inverse quasi-variational inequality problems. Using the Lyapunov theory functional differential equations, we have established, under certain conditions, the existence of the solution to the proposed network, as well as its asymptotic stability and exponential stability.
- We have proved that the sequence generated by the discretization of the network (10) converges to the solution of the IQVIP (8) under certain assumptions on the parameters involved. Finally, we have provided a numerical example to illustrate our theoretical analysis.

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THANK YOU