On Krasnoselškii fixed point theorems

Tian Xiang

(Institute for Mathematical Sciences, Renmin University of China)

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Outline

1. Krasnsoselškii's fixed point theorem and extensions
   - Extending contraction to other types
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   - General extensions of Krasnosel'skii fixed point theorem
2. Krasnsoselškii cone compression-expansion fixed point theorem
   - Extensions of cone fixed point theorems
   - Cone fixed point theorems for sum operators
3. Indicative applications of Krasnsoselškii's fixed point theorems

Krasnsoselškii fixed point theorem

In 1958, Krasnsoselškii observed that the inversion of a perturbed differential operator may yield the sum of a compact and a contraction operator, i.e., $F = T + S$. Then he combined the well-known Banach contraction mapping principle and Schauder's fixed point theorem to obtain the following celebrated fixed point theorem:

Theorem (Krasnsoselškii fixed point theorem)

Let K be a nonempty closed convex subset of a Banach space $(E, \|\cdot\|)$. Assume that T and S map K into E such that

(K1) *T is an $\alpha \in [0,1)$-contraction: $\|Tx - Ty\| \leq \alpha \|x - y\|$;*

(K2) *S is continuous and $S(K)$ is pre-compact in E;*

(K3) *any $x, y \in K$ imply $Tx + Sy \in K$.*

Then $T + S$ has at least one fixed point $x^* \in K$, i.e., $Tx^* + Sx^* = x^*$.

Remarks on Krasnsoselškii fixed point theorem

The sum operator $F = T + S$ is neither compact, monotone or contractive, on which fixed point theory and nonlinear functional analysis usually based. The original KFPT has been generalized in various directions from theory- or application-based perspectives.

Main directions of generalizations of KFPT

(1) Extending the contraction of T to other types, ...;
(2) Relaxing the compactness of S, ...;
(3) Relaxing the condition (K3): "any $x, y \in K$ imply $Tx + Sy \in K$";
(4) Extending the underlying space: weak or strong or locally convex;
(5) Extending to muti-valued mappings;
(6) Extending to cone fixed point theorems;
(7) Extending to nonlinear combination of T, S, i.e., $x = F(Tx, Sx)$....,

I'd like to select applied-oriented versions of generalizations.

Bases for generalizations of KFPT

$$x = Tx + Sx \Longleftrightarrow x \in (I - T)^{-1}Sx \Longleftrightarrow x \in S^{-1}(I - T)x.$$

Strong topology: general version of Schauder FFT, Sadovskii:

$$N : K(bdd) \to K \text{ is condensing } \implies N \text{ has a fixed point in } K.$$

Strong topology: multi-valued map(Zeidler, X.-Yuan, PAMS, 09'):

$$F : K \to 2^K \text{condensing, usc } + \text{cc range} \Longrightarrow \exists x \in K \ s.t. \ x \in Fx.$$

Bases for generalizations of KFPT, cont.

Weak topology: (Garcia-Falset, NA, 09'):

$$N : K \to K \text{ is w-condensing+wsc} \Longrightarrow \exists x \in K \ s.t. \ x = Nx.$$

Weak topology: multi-valued (Arino-Gautier-Penot, Ben Amar-X.)

$$F : K \to 2^K \text{is w-condensing, wsc+cc+bdd range} \Longrightarrow \exists x \in K \ s.t. \ x \in Fx.$$

Countably condensing and countably weakly condensing: (Daher, 78') for single-valued map; Himmelberg, Agarwal, ÒRegan, Cardinali, Ben Amar, X. etc for usc multi-valued maps.

Extending the contraction of T to other types

- T is a Bond-Wang contraction: $\|Tx - Ty\| \leq \phi(\|x-y\|)$ with $\phi(t) < t$ and right continuous (Nashed-Wong, 96', JMM).
- Meir-Keeler (96') contraction: $\forall \epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \|x-y\| < \epsilon + \delta \Longrightarrow \|Tx - Ty\| \leq \epsilon.$$

  cf. Sehgal-Sing, Hoa and Schmitt, Dhage, Garcia-Falset, etc.
- T is nonexpansive, Bruck, Browder, Agarwal, O' Regan, Taoudi etc.
- T is expansive: $\|Tx - Ty\| \geq h\|x-y\|$, (Avramescu- Vladimirescu, 03', FPT) and (X.-Yuan, 09', NA) with applications to integral equations. Benzenati, Mebarki, etc.
- T is non-linearly expansive, $\|Tx - Ty\| \geq \phi(\|x-y\|)$ with $\phi(t) > t$ and right continuous; with application to IE (Wang-Wang, 12', FFT).

Relaxing (K3): "$x, y \in K$ imply $Tx + Sy \in K$"

- Burton (98', AML) relaxed (K3) to the following widely used one:

$$\{[x = Tx + Sy,\ y \in K] \Longrightarrow x \in K\} \Longrightarrow (I - T)^{-1}S : K \to K.$$

  See also [Barroso, 03', NA], where $T \in \mathcal{L}(E)$, $\|T^p\| \leq 1$ for some $p \geq 1$, $\lambda \in (0,1)$, $T \mapsto \lambda T$ and $S \mapsto \lambda S$.

- Based on Rothe's theorem, Chen [20', JFPTA] replaced (K3) by

$$\left\{[x = Tx + Sy,\ y \in \partial G] \Longrightarrow x \in \overline{G}\right\} \Longrightarrow (I - T)^{-1}S(\partial G) \subset \overline{G}$$

- In a series of our works [PAMS, MMAS, JFPTA], we work on the set

$$\mathcal{F} = \mathcal{F}(M, K; T, S) = \{x \in M : x = Tx + Sy \text{ for some } y \in K\}.$$

General extensions of KFPT in strong topology

- 17 extensions when S is compact and $(I-T)^{-1}$ is continuous on $\overline{S(K)}$ and K has the compact fixed point property, [Park, 07',NA'].
- S is compact and $(I-T)$ may not be injective, convex and proper (compact) assumption on $(I-T)(K)$, [Liu-Li, 08', PAMS].
- S is noncompact, $(I-T)$ may not be injective and be not necessarily continuous, applications to neutral ODEs [X.-Yuan, 11', PAMS].
- S is not necessarily compact and T is not necessarily continuous [Djebali, Mebarki etc] and [X.-Georgiev, 16', MMAS] with applications to transport equations, Darboux problems, difference equations and Volterra-type integral equations.
- T is dissipative and S is condensing [X., 11'; X.-Georgiev, 16'].
- Critical type Krasnosel'skii fixed point theorems [X.-Yuan, 15', FPTA].

General extensions of KFPT in weak topology

- S is wsc and E is lctvs [Barroso-Teixeira, 05', NA], PDE applications.
- On Dunford-Pettis spaces and appl.s[Ben Amar et.al., 05', M2AS].
- S and T satisfy some ws (may not be wc) and ww conditions and applications to transport equations in L^1 [Latrach, Taoudi, Zeghal, Garcia-Falset, Djebali, Sahnoun, Cai-Bu etc].
- S is wc and T may not be continuous, Banaś, X.-Yuan, Taoudi etc.
- S is weakly noncompact via MWNC, Taoudi-X., Garcia-Falset, Ezzinbi etc.
- Critical type KFPT in weak topology [Ben Amar-X., 15', QM; Ben Amar et.al, 16',FPT].

Extensions of KFPT to multi-valued maps

- Invariant sets without convexity [Ok, 09', PAMS].
- S is multi-valued [Graef, Henderson et al., 17', JFPTA].
- Perturbed ws/ww multi-valued maps [Latrach et al., 21',JFPTA].
- Fixed point set in weak topology [Ben Amar et.al., 17', FPT].
- In Banach algebra [Dhage, Ben Amar, O'Regan etc].
- T and S are set-valued [Plubtieng, Basoc, Cardinali etc].
- Weakly sequentially closed graph for S [Ben Amar, O'Regan etc].
- Krasnosel'skii-Schaefer type [Dhage, Burton, Kirk, X.-Yuan, etc].
- Multi-valued nonexpansive maps [Bounegab-Djebali, 19', MJM].

Extensions to cone/positive fixed point theorems

- Krasnoselskii-Petryshyn compression and expansion for single countably condensing map [Agarwal-O'Regan, 01',JKMS]
- Krasnoselskii cone expansion-compression of functional type with applications [Anderson, Avery, Kwong, Zhang, Sun, O'Regan, etc].
- Krasnoselskii cone compression theorems for other 'compact' multi-maps [O'Regan, Shahzad, Agarwal etc].
- Cone compression and expansion fixed point theorems for multi-valued maps in Fréchet spaces with applications [Agarwal, O'Regan, Frigon etc].
- Cone fixed points of norm/functional type for expansive maps and k-set contractions [Benzenati-Mebarki,19', MMAS].

Extension to positive fixed point theorems, II

- Cone compression-expansion FPT of norm type [Guo, Petryshyn, O'Regan, Precup, etc] .
- Vector version of cone fixed point theorem [Precup, 07, JFPTA]
- Functional expansion-compression fixed point theorem of Leggett-Williams type [Anderson, Avery, Henderson, Liu et.al.]
- User-friendly versions of fixed point theorem in cones of norm type for $T + S$, with T contractive/expansive and S compact/set-contractive with various applications [X.-Zhu,'20, JFPTA].
- Leggett-Williams type fixed point theorem for sums of operators with applications [Georgiev-Mebarki, 21' DEA]

Cone fixed point theorem for $T + S$

Theorem (Cone compression-expansion FPT, X.-Zhu, 20', JFPTA)

Suppose that $S : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to E$ *is completely continuous and* $T : E \to E$ *is contractive with constant* $\alpha < 1$ *such that*

(i) $[x = Tx + Sy, y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)] \Longrightarrow x \in P$;

in addition, suppose either one of the following conditions is satisfied:

(ii) $\|Sx + T\theta\| \leq (1-\alpha)\|x\|, \forall x \in P \cap \partial\Omega_1$, *and*
$\|Sx + T\theta\| \geq (1+\alpha)\|x\|, \forall x \in P \cap \partial\Omega_2$;

(iii) $\|Sx + T\theta\| \geq (1+\alpha)\|x\|, \forall x \in P \cap \partial\Omega_1$, *and*
$\|Sx + T\theta\| \leq (1-\alpha)\|x\|, \forall x \in P \cap \partial\Omega_2$.

Then there exists $x^* \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ *with* $Tx^* + Sx^* = x^*$.

Remark: S can be relaxed as strict $(1-\alpha)$-set contractive.

Cone fixed point theorem for $T+S$

Theorem (Cone compression-expansion FPT, X.-Zhu, 20', JFPTA)

$S : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to E$ *is completely continuous (strict* $(h-1)$ *set-contractive) and* $T : E \to E$ *is h-expansive and l-Lipschitz such that*

(i) $S(P \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset (I - T)(E)$ *and*
$[x = Tx + Sy, y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)] \Longrightarrow x \in P$;

besides, assume either one of the following conditions is fulfilled:

(ii) $\|Sx + T\theta\| \leq (h-1)\|x\|, \forall x \in P \cap \partial\Omega_1$, *and*
$\|Sx + T\theta\| \geq (l+1)\|x\|, \forall x \in P \cap \partial\Omega_2$;

(iii) $\|Sx + T\theta\| \geq (l+1)\|x\|, \forall x \in P \cap \partial\Omega_1$, *and*
$\|Sx + T\theta\| \leq (h-1)\|x\|, \forall x \in P \cap \partial\Omega_2$.

Then $T + S$ *possesses at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Indicative applications of KFPTs

- (Functional) Integral equations of Hammerstein and perturbed Volterra type [Burton etc]
- Transport equations in L^1 and L^p.
- Elliptic PDEs [Barroso etc]
- Delay differential equations
- Neutral differential equations
- Evolution equations [Ezzinbi, Taoudi etc]
- Navier-Stokes equations [Zennir, Georgiev etc]
- Parabolic and wave equations [Georgiev etc]
- Lotka-Volterra competition systems [Tang, Zhou etc]
- BVP for 2nd ODEs
- Periodic solutions in dynamical systems [Torres, Wang etc]

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