On Krasnosel’skii fixed point theorems

Tian Xiang

(Institute for Mathematical Sciences, Renmin University of China)

Workshop on Nonlinear Functional Analysis and Its Applications in memory of Prof. Ronald E. Bruck
Israel Institute of Technology

April 4-6, 2022.
Outline

1. Krasnosel’skii’s fixed point theorem and extensions
   - Extending contraction to other types
   - Relaxing contraction set conditions
   - General extensions of Krasnosel’skii fixed point theorem

2. Krasnosel’skii cone compression-expansion fixed point theorem
   - Extensions of cone fixed point theorems
   - Cone fixed point theorems for sum operators

3. Indicative applications of Krasnosel’skii’s fixed point theorems
Krasnosel’skii fixed point theorem

In 1958, Krasnosel’skii observed that the inversion of a perturbed differential operator may yield the sum of a compact and a contraction operator, i.e., $F = T + S$. Then he combined the well-known Banach contraction mapping principle and Schauder’s fixed point theorem to obtain the following celebrated fixed point theorem:

**Theorem (Krasnosel’skii fixed point theorem)**

Let $K$ be a nonempty closed convex subset of a Banach space $(E, \| \cdot \|)$. Assume that $T$ and $S$ map $K$ into $E$ such that

(K1) $T$ is an $\alpha \in [0, 1)$-contraction: $\|Tx - Ty\| \leq \alpha \|x - y\|$

(K2) $S$ is continuous and $S(K)$ is pre-compact in $E$

(K3) any $x, y \in K$ imply $Tx + Sy \in K$

Then $T + S$ has at least one fixed point $x^* \in K$, i.e., $Tx^* + Sx^* = x^*$.
Remarks on Krasnosel’skii fixed point theorem

The sum operator $F = T + S$ is neither compact, monotone or contractive, on which fixed point theory and nonlinear functional analysis usually based. The original KFPT has been generalized in various directions from theory- or application-based perspectives.

**Main directions of generalizations of KFPT**

1. Extending the contraction of $T$ to other types, ....;
2. Relaxing the compactness of $S$, .....;
3. Relaxing the condition (K3): "any $x, y \in K$ imply $Tx + Sy \in K$";
4. Extending the underlying space: weak or strong or locally convex;
5. Extending to multi-valued mappings;
6. Extending to cone fixed point theorems;
7. Extending to nonlinear combination of $T, S$, i.e., $x = F(Tx, Sx).....,$

I’d like to select applied-oriented versions of generalizations.
Bases for generalizations of KFPT

\[ x = Tx + Sx \iff x \in (I - T)^{-1}Sx \iff x \in S^{-1}(I - T)x. \]

**Strong topology:** general version of Schauder FFT, Sadovskii:

\[ N : K(bdd) \to K \text{ is condensing} \implies N \text{ has a fixed point in } K. \]

**Strong topology:** multi-valued map (Zeidler, X.-Yuan, PAMS, 09’):

\[ F : K \to 2^K \text{ condensing, usc +cc range} \implies \exists x \in K \text{ s.t. } x \in Fx. \]
Bases for generalizations of KFPT, cont.

**Weak topology**: (Garcia-Falset, NA, 09'):

\[ N : K \to K \text{ is } w\text{-condensing } + \text{wsc } \implies \exists x \in K \text{ s.t. } x = Nx. \]

**Weak topology**: multi-valued (Arino-Gautier-Penot, Ben Amar-X.)

\[ F : K \to 2^K \text{ is } w\text{-condensing, wsc+cc+bdd range } \implies \exists x \in K \text{ s.t. } x \in Fx. \]

**Countably condensing and countably weakly condensing**: (Daher, 78') for single-valued map; Himmelberg, Agarwal, ÒRegan, Cardinali, Ben Amar, X. etc for usc multi-valued maps.
Extending the contraction of $T$ to other types

- $T$ is a Bond-Wang contraction: $\|Tx - Ty\| \leq \phi(\|x - y\|)$ with $\phi(t) < t$ and right continuous (Nashed-Wong, 96', JMM).
- Meir-Keeler (96') contraction: $\forall \epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \|x - y\| < \epsilon + \delta \implies \| Tx - Ty \| \leq \epsilon.$$ 

cf. Sehgal-Sing, Hoa and Schmitt, Dhage, Garcia-Falset, etc.

- $T$ is nonexpansive, Bruck, Browder, Agarwal, O’ Regan, Taoudi etc.
- $T$ is expansive: $\|Tx - Ty\| \geq h\|x - y\|$, (Avramescu- Vladimirescu, 03’, FPT) and (X.-Yuan, 09’, NA) with applications to integral equations. Benzenati, Mebarki, etc.
- $T$ is non-linearly expansive, $\|Tx - Ty\| \geq \phi(\|x - y\|)$ with $\phi(t) > t$ and right continuous; with application to IE (Wang-Wang, 12’, FFT).
Relaxing (K3): "\( x, y \in K \) imply \( Tx + Sy \in K \)"

- Burton (98', AML) relaxed (K3) to the following widely used one:

\[
\{ [x = Tx + Sy, \ y \in K] \implies x \in K \} \implies (I - T)^{-1}S : K \to K.
\]

See also [Barroso, 03', NA], where \( T \in \mathcal{L}(E), \|T^p\| \leq 1 \) for some \( p \geq 1, \lambda \in (0, 1), T \mapsto \lambda T \) and \( S \mapsto \lambda S \).

- Based on Rothe’s theorem, Chen [20’, JFPTA] replaced (K3) by

\[
\{ [x = Tx + Sy, \ y \in \partial G] \implies x \in \overline{G} \} \implies (I - T)^{-1}S(\partial G) \subset \overline{G}
\]

- In a series of our works [PAMS, MMAS, JFPTA], we work on the set

\[
\mathcal{F} = \mathcal{F}(M, K; T, S) = \{ x \in M : x = Tx + Sy \text{ for some } y \in K \}.
\]
General extensions of KFPT in strong topology

- 17 extensions when $S$ is compact and $(I - T)^{-1}$ is continuous on $S(K)$ and $K$ has the compact fixed point property, [Park, 07’, NA’].
- $S$ is compact and $(I - T)$ may not be injective, convex and proper (compact) assumption on $(I - T)(K)$, [Liu-Li, 08’, PAMS].
- $S$ is noncompact, $(I - T)$ may not be injective and be not necessarily continuous, applications to neutral ODEs [X.-Yuan, 11’, PAMS].
- $S$ is not necessarily compact and $T$ is not necessarily continuous [Djebali, Mebarki etc] and [X.-Georgiev, 16’, MMAS] with applications to transport equations, Darboux problems, difference equations and Volterra-type integral equations.
- $T$ is dissipative and $S$ is condensing [X., 11’; X.-Georgiev, 16’].
- Critical type Krasnosel’skii fixed point theorems [X.-Yuan, 15’, FPTA].
General extensions of KFPT in weak topology

- $S$ is wsc and $E$ is lctvs [Barroso-Teixeira, 05’, NA], PDE applications.
- On Dunford-Pettis spaces and appl.s[Ben Amar et.al., 05’, M2AS].
- $S$ and $T$ satisfy some ws (may not be wc) and ww conditions and applications to transport equations in $L^1$ [Latrach, Taoudi, Zeghal, Garcia-Falset, Djebali, Sahnoun, Cai-Bu etc].
- $S$ is wc and $T$ may not be continuous, Banaś, X.-Yuan, Taoudi etc.
- $S$ is weakly noncompact via MWNC, Taoudi-X., Garcia-Falset, Ezzinbi etc.
- Critical type KFPT in weak topology [Ben Amar-X., 15’, QM; Ben Amar et.al, 16’, FPT].
Extensions of KFPT to multi-valued maps

- Invariant sets without convexity [Ok, 09’, PAMS].
- $S$ is multi-valued [Graef, Henderson et al., 17’, JFPTA].
- Perturbed ws/ww multi-valued maps [Latrach et al., 21’, JFPTA].
- Fixed point set in weak topology [Ben Amar et al., 17’, FPT].
- In Banach algebra [Dhage, Ben Amar, O’Regan etc].
- $T$ and $S$ are set-valued [Plubtieng, Basoc, Cardinali etc].
- Weakly sequentially closed graph for $S$ [Ben Amar, O’Regan etc].
- Krasnosel’skii-Schaefer type [Dhage, Burton, Kirk, X.-Yuan, etc].
- Multi-valued nonexpansive maps [Bounegab-Djebali, 19’, MJM].
Extensions to cone/positive fixed point theorems

- Krasnoselskii-Petryshyn compression and expansion for single countably condensing map [Agarwal-O’Regan, 01’, JKMS]
- Krasnoselskii cone expansion-compression of functional type with applications [Anderson, Avery, Kwong, Zhang, Sun, O’Regan, etc].
- Krasnoselskii cone compression theorems for other ‘compact’ multi-maps [O’Regan, Shahzad, Agarwal etc].
- Cone compression and expansion fixed point theorems for multi-valued maps in Fréchet spaces with applications [Agarwal, O’Regan, Frigon etc].
- Cone fixed points of norm/functional type for expansive maps and \( k \)-set contractions [Benzenati-Mebarki, 19’, MMAS].
Extension to positive fixed point theorems, II

- Cone compression-expansion FPT of norm type [Guo, Petryshyn, O’Regan, Precup, etc].
- Vector version of cone fixed point theorem [Precup, 07, JFPTA]
- Functional expansion-compression fixed point theorem of Leggett-Williams type [Anderson, Avery, Henderson, Liu et.al.]
- User-friendly versions of fixed point theorem in cones of norm type for $T + S$, with $T$ contractive/expansive and $S$ compact/set-contractive with various applications [X.-Zhu,’20, JFPTA].
- Leggett-Williams type fixed point theorem for sums of operators with applications [Georgiev-Mebarki, 21’ DEA]
Theorem (Cone compression-expansion FPT, X.-Zhu, 20’, JFPTA)

Suppose that $S : P \cap (\Omega_2 \setminus \Omega_1) \to E$ is completely continuous and $T : E \to E$ is contractive with constant $\alpha < 1$ such that

(i) $[x = Tx + Sy, \ y \in P \cap (\Omega_2 \setminus \Omega_1)] \implies x \in P$;

in addition, suppose either one of the following conditions is satisfied:

(ii) $\|Sx + T\theta\| \leq (1 - \alpha)\|x\|$, $\forall x \in P \cap \partial \Omega_1$, and

$\|Sx + T\theta\| \geq (1 + \alpha)\|x\|$, $\forall x \in P \cap \partial \Omega_2$;

(iii) $\|Sx + T\theta\| \geq (1 + \alpha)\|x\|$, $\forall x \in P \cap \partial \Omega_1$, and

$\|Sx + T\theta\| \leq (1 - \alpha)\|x\|$, $\forall x \in P \cap \partial \Omega_2$.

Then there exists $x^* \in P \cap (\Omega_2 \setminus \Omega_1)$ with $Tx^* + Sx^* = x^*$.

Remark: $S$ can be relaxed as strict $(1 - \alpha)$-set contractive.
Cone fixed point theorem for $T + S$

**Theorem (Cone compression-expansion FPT, X.-Zhu, 20', JFPTA)**

$S : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to E$ is completely continuous (strict $(h-1)$ set-contractive) and $T : E \to E$ is $h$-expansive and $l$-Lipschitz such that

(i) $S(P \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset (I - T)(E)$ and

$$[x = Tx + Sy, y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)] \implies x \in P;$$

besides, assume either one of the following conditions is fulfilled:

(ii) $\|Sx + T\theta\| \leq (h-1)\|x\|, \forall x \in P \cap \partial\Omega_1$, and

$$\|Sx + T\theta\| \geq (l+1)\|x\|, \forall x \in P \cap \partial\Omega_2;$$

(iii) $\|Sx + T\theta\| \geq (l+1)\|x\|, \forall x \in P \cap \partial\Omega_1$, and

$$\|Sx + T\theta\| \leq (h-1)\|x\|, \forall x \in P \cap \partial\Omega_2.$$

Then $T + S$ possesses at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. 
Indicative applications of KFPTs

- (Functional) Integral equations of Hammerstein and perturbed Volterra type [Burton etc]
- Transport equations in $L^1$ and $L^p$.
- Elliptic PDEs [Barroso etc]
- Delay differential equations
- Neutral differential equations
- Evolution equations [Ezzinbi, Taoudi etc]
- Navier-Stokes equations [Zennir, Georgiev etc]
- Parabolic and wave equations [Georgiev etc]
- Lotka-Volterra competition systems [Tang, Zhou etc]
- BVP for 2nd ODEs
- Periodic solutions in dynamical systems [Torres, Wang etc]