The influence of the work of R.E. Bruck on the development of proof mining

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Nonlinear Functional Analysis and Its Applications in memory of Professor Ronald E. Bruck, via Zoom, Technion Haifa April 4-6, 2022

Proof Mining in core mathematics

- During (mainly) the last 20 years method from mathematical logic (proof theory) were used to obtain **numerous new quantitative results** as well as **qualitative uniformity results** in particular in: nonlinear analysis, fixed point theory, ergodic theory, topological dynamics, approximation theory, convex optimization, abstract Cauchy problems, pursuit-evasion games (≥ 100 papers mostly in specialized journals in the resp. areas or general mathematics journals).

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- General **logical metatheorems** explain applications as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).
- Some of the logical tools used have been rediscovered in 2007 in special cases by Terence Tao prompted by concrete mathematical needs **"finitary analysis"!**

Logical Metatheorems based on Functional Interpretations

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In case of a restricted use of noneffective reasoning, A can have arbitrary logical complexity.

The running theme: convergence statements in analysis

Let (x_n) be a Cauchy sequence in a metric space (X, d), i.e.

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \geq n\, (d(x_i, x_j) \leq 2^{-k}) \in \forall\exists\forall$$

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A bound $\Phi(k, g)$ on '$\exists n$' in the latter formula is a **rate of metastability**.

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- Often **possible for asymptotic regularity** results

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- Possible also in the nonunique case for **Fejér monotone algorithms** if one has a **modulus of metric regularity** (see K./López-Acedo/Nicolae Israel J. Math. 2019).

- **Admissible abstract structures:** metric, hyperbolic, CAT(0), CAT($\kappa > 0$), Ptolemy, normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces, abstract L^p- and $C(K)$-spaces (and all normed structures axiomatizable in positive bounded logic (in the sense of Henson, Iovino, Ben-Yaacov etc.).

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Bruck's research and proof mining

- Various classes of mappings introduced by Bruck (firmly nonexpansive and - with Reich - averaged and strongly (quasi-) nonexpansive mappings are logically very well-behaved in the aforementioned metatheorems and play an important role in proof mining (see below).

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- Bruck's work on the convex approximation property and nonlinear ergodic theorems recently has been investigated from the perspective of proof mining (see below).

An example for proof mining involving strongly nonexpansive mappings

A polynomial rate of asymptotic regularity in Bauschke's solution of the 'zero displacement conjecture'

Consider a Hilbert space H and nonempty closed and convex subsets $\boldsymbol{C_1, \ldots, C_N \subseteq H}$ with metric projections $\boldsymbol{P_{C_i}}$, define $\boldsymbol{T := P_{C_N} \circ \ldots \circ P_{C_1}}$. In 2003 Bauschke proved the 'zero displacement conjecture':

$$\boldsymbol{\|T^{n+1}x - T^n x\| \to 0 \quad (x \in H).}$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^{N} C_i \neq \emptyset$) or C_i half spaces etc.

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Logical metatheorems guarantee an effective rate of convergence which only depends on $\varepsilon, N, b \geq \|x\|, K \geq \|c_i\|$ for some $c_i \in C_i$ $(1 \leq i \leq N)$.

Theorem (K. FoCM 2019)

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

is a **rate of asymptotic regularity** in Bauschke's result, where

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, \quad D := 2b + NK, \quad \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(K^2 + N^3(N-1)^2K^2)N^2}{\varepsilon}.$$

Here $b \geq \|x\|$ and $K \geq \left(\sum_{i=1}^{N} \|c_i\|^2\right)^{\frac{1}{2}}$ for some $(c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N$.

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A. Sipoș: Extension to general **averaged mappings**. Opt.Lett. 2021.

Applications to Ergodic Theory

Let X be a uniformly convex Banach space, $C \subseteq X$ nonempty, bounded, closed and convex, $T : C \to C$ nonexpansive.

$$S_n x := \frac{1}{n} \sum_{i=0}^{n-1} T^i x.$$

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Theorem (Bruck 1981)

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Bruck's proof crucially uses a result due to Pisier 1973 that uniformly convex Banach spaces (in fact uniformly nonsquare and even B-convex Banach spaces) have a nontrivial Rademacher type $q > 1$ which implies that there exists a constant C_q s.t. for every finite sequence $X_1, \ldots, X_n$ of independent mean zero Radon random variables in $L_q(X)$

$$\mathbb{E}\left(\left\|\sum_{i=1}^{n} X_i\right\|^q\right)^{1/q} \leq 2C_q \cdot \left(\sum_{i=1}^{n} \mathbb{E}\left(\|X_i\|^q\right)\right)^{1/q}.$$

Let $(X, \|\cdot\|)$ be **uniformly nonsquare** with $\delta \in (0,1]$ such that

$$\min\left\{\frac{\|x-y\|}{2}, \frac{\|x+y\|}{2}\right\} \leq (1-\delta)\cdot\max\{\|x\|,\|y\|\}$$

holds for all $x, y \in X$.

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If X is uniformly convex and η is a modulus of convexity, we may take $\delta := \eta(1)$.

From Pisier's proof one can extract the following explicit constants:

Theorem (Pisier 1973, Freund/K. 2022 submitted)

Let $(X, \|\cdot\|)$ be a uniformly nonsquare Banach space with $\delta \in (0,1)$ witnessing this property. Define $\lambda := 1-\delta$. Assume that $\xi \in (0,1)$ is so small and that $p' \in [2,\infty)$ is so large that

$$\frac{1-\xi}{1+2\sqrt{2\xi}} \geq \frac{1}{2}\sqrt{2\lambda^2+2} \quad \text{and} \quad \frac{1}{2^{1/p'}} \geq 1-\xi.$$

Take p with $1 = \frac{1}{p} + \frac{1}{p'}$.

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Take p with $1 = \frac{1}{p} + \frac{1}{p'}$.
Then - for any $q \in (1,p)$ - X has Rademacher type q with constant

$$C_q = 3 \cdot \frac{2^{1/q}}{2^{(1/q)-(1/p)} - 1}.$$

Analyzing proofs due to Bruck 1979, 1981 one can extract:

Theorem (Freund/K. 2022 submitted)

Let $(X, \|\cdot\|)$ be uniformly convex with modulus η. $T : C \to C$ nonexpansive, $C \subseteq B_{b/2}(0)$ closed and convex. Assume X is of Rademacher type $q \in (1,2]$ with constant C_q. Given $\varepsilon > 0$, pick $\tilde{p} \in \mathbb{N}$ so large that $2C_q \cdot \tilde{p}^{(1-q)/q} \leq \varepsilon/(9b)$. Consider $p \in \mathbb{N}$ with $p \geq 2b/\delta^2$ for

$$\delta := \xi^{\tilde{p}}\left(\frac{\varepsilon}{9}\right) \quad \text{with} \quad \xi(t) := \frac{t}{12} \cdot \eta\left(\min\left\{2, \frac{t}{b}\right\}\right).$$

For any $\alpha < \xi^{p-1}(\delta^2/2)$ with $0 < \alpha < \varepsilon/3$ we have

$$\forall n \geq \frac{b}{\alpha} \ \forall x \in C \ (\|T(S_n x) - S_n x\| \leq \varepsilon).$$

Theorem (Kobayasi-Miyadera 1980)

Consider $T : C \to C$ as above. Given $x \in C$, assume that the sequences $(\|T^n x - T^{n+i} x\|)_n$ converge uniformly in i. We then have

$$\lim_{n\to\infty} \|y - S_n T^k x\| = 0 \quad \text{uniformly in } k,$$

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The proof by Kobayasi and Miyadera makes crucial use of Bruck's work on nonlinear ergodic theory and the convex approximation property 1979-81.

The proof actually shows that $(S_m T^m x)$ converges to y in the theorem and the complete conclusion can be decomposed into:

$$\lim_{m,n\to\infty} \left\| S_m T^m x - S_n T^k x \right\| = 0 \quad \text{uniformly in } k, \tag{1}$$

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$$\lim_{n\to\infty} \|T S_n T^k x - S_n T^k x\| = 0 \quad \text{uniformly in } k. \tag{3}$$

We extract an explicit function Φ which translates any rate A of metastability for the assumption that the sequences $(\|T^n x - T^{n+i} x\|)_n$ converge uniformly in i into a common rate of metastability for (1)-(3) :

A quantitative Kobayasi-Miyadera theorem

$$\alpha_n^i := \|T^n x - T^{n+i} x\|.$$

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Theorem (Freund/K. ETDS 2022)

Let η be a modulus of uniform convexity for X. Let $C \subseteq B_b(0)$ and -for $x \in C$ - A be such that for all $\varepsilon > 0$ and $h : \mathbb{N} \to \mathbb{N}$

$$\exists N \leq A(\varepsilon, g, h)\, \forall m, n \in [N, N+g(N)], i \leq h(N)\, (|\alpha_m^i - \alpha_n^i| < \varepsilon).$$

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We get an effective bound $\exists N \leq \Phi(A, b, \eta, \varepsilon, g, h)$ with

$$\left\|S_m T^m x - S_n T^k x\right\|, \left\|T^l S_n T^n x - S_n T^n x\right\|, \left\|T^l S_n T^k x - S_n T^k x\right\| < \varepsilon$$

for all $m, n \in [N, N+g(N)]$ and $k, l \leq h(N)$.

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- X is **uniformly convex** and C (or just $\{T^n x : n \in \mathbb{N}\}$) is **compact**. Then A additionally depends on a **modulus of total boundedness**; Freund/K. ETDS 2022.