Modular approximation in convergence spaces

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Abstract

In this talk, we will investigate the latest approximation results obtained in modular spaces equipped with a convergence structure. A class of such spaces contains all Banach spaces, modular function spaces such as Lebesgue, Orlicz spaces, variable Lebesgue spaces, and their generalisations. We will discuss questions of existence and uniqueness of best approximants in the sense of modular distances in such spaces. In this context, we will also investigate the continuity aspects of generally nonlinear, multi-valued modular projection operators. We will indicate some important areas of application.

Modular spaces - Definitions

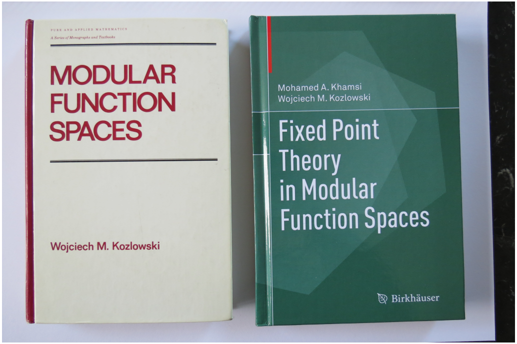
Modular Spaces (Nakano, 1950; Musielak and Orlicz, 1959

Let X be a real vector space. A functional $\rho : X \to [0, \infty]$ is called a convex modular if

1. $\rho(x) = 0$ if and only if $x = 0$
2. $\rho(-x) = \rho(x)$
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for any $x, y \in X$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

Convex modular, when (3) is replaced by $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$. The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0,\ as\ \lambda \to 0\}$ is called a modular space.

Examples of modulars:
$\rho(x) = \int_\Omega |x(t)|^{p(t)} d\mu(t),\ \int_\Omega \varphi(x(t)) d\mu(t),\ \int_\Omega \varphi(x(t), t) d\mu(t)$
Generalisation: abstract concepts of Function Modulars and Modular Function Spaces (MFS).
Extensive applications of MFS, particularly in Fixed Point Theory.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ρ-Convergence in modular spaces - Definitions

ρ-convergence

- (a) $x_n \xrightarrow{\rho} x$ (ρ-convergent to x) if $\rho(x_n - x) \to 0$.
- (b) $\{x_n\}$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$. X_ρ is called ρ-complete if every ρ-Cauchy is ρ-convergent to an $x \in X_\rho$.
- (c) A set $B \subset X_\rho$ is called ρ-closed if for any sequence of $x_n \in B$, the convergence $x_n \xrightarrow{\rho} x$ implies that x belongs to B.
- (d) A set $B \subset X_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B) = \sup\{\rho(x - y) : x \in B, y \in B\}$ is finite.
- (e) A set $K \subset X_\rho$ is called ρ-compact if for any $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.
- (f) Let $x \in X_\rho$ and $C \subset X_\rho$. The ρ-distance between x and C is defined as $d_\rho(x, C) = inf\{\rho(x - y) : y \in C\}$.
- (g) A ρ-ball $B_\rho(x, r)$ is defined by $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \le r\}$.

L-Convergence - Definitions

L-convergence (Fréchet 1918, Urysohn 1926, Kisyński 1959)

A relation ζ between sequences $\{x_n\}_{n=1}^{\infty}$ of elements of X and elements x of X, denoted by $x_n \xrightarrow{\zeta} x$, is called L-convergence on X if

1. if $x_n = x$ for all $n \in \mathbb{N}$ then $x_n \xrightarrow{\zeta} x$,
2. if $x_n \xrightarrow{\zeta} x$ and $\{x_{n_k}\}$ is a proper subsequence of $\{x_n\}$, then $x_{n_k} \xrightarrow{\zeta} x$,
3. if $x_n \xrightarrow{\zeta} x$ and $x_n \xrightarrow{\zeta} y$, then $x = y$.

The pair (X, ζ) (or shortly X) is called an L-convergence space, or shortly L-space.

Notes: ρ-convergence is an example of L-convergence. Closed, open and sequentially compact sets can be defined in the same way. Convergence in any topology is an L-convergence. Convergence almost everywhere is an example of an L-convergence which is not generated by any topology.

Combining L-spaces with modular spaces - Definitions

Modulated LTI-spaces (Kozlowski, JAT 2021)

Let ρ be a modular defined on X and let ζ be an L-convergence on X_ρ. The triplet (X_ρ, ρ, ζ) is called a modulated LTI-space if (X_ρ, ζ) is an L-space and the following three conditions are satisfied

1. (i) $x_n \xrightarrow{\zeta} x$ implies that $x_n - y \xrightarrow{\zeta} x - y$ for any $y \in X_\rho$
2. (ii) $x_n \xrightarrow{\zeta} x \Rightarrow \rho(x) \le \liminf_{n\to\infty} \rho(x_n)$,
3. (iii) if $x_n \xrightarrow{\rho} x$ then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\zeta} x$, where x, $x_n \in X$.

- (a) Every ζ-closed set is also ρ-closed.
- (b) Every ρ-compact set is also sequentially ζ-compact.
- (c) Every ρ-ball $B_\rho(x, r)$ is ζ-closed (and hence also ρ-closed).

Best approximants in (LTI)-spaces

Examples of (LTI)-spaces: Banach spaces, Modular function spaces, Orlicz spaces, Lebesgue spaces, variable Lebesgue spaces.

Existence and Uniqueness Theorem A (Kozlowski, JAT 2021)

Let (X_ρ, ρ, ζ) be a modulated, ρ-complete LTI-space, where ρ is a convex modular satisfying $(UUC2)$. Assume that $C \subset X_\rho$ is nonempty, convex and ρ-closed. Let $x \in X_\rho$ be such that $d_\rho(x, C) = \inf_{y \in C} \rho(x - y) < \infty$.
There exists a unique $x_0 \in C$ such that $\rho(x - x_0) = d_\rho(x, C)$.
Such element x_0 is called the best ρ-approximant of x with respect to C.

We say that ρ satisfies $(UUC2)$ (a form of a modular uniform convexity) if for every $s \geq 0,\ \varepsilon > 0$ there exists $\eta_2(s, \varepsilon) > 0$ such that
$\delta_2(r, \varepsilon) > \eta_2(s, \varepsilon) > 0 \ for \ r > s$, where $\delta_2(r, \varepsilon) =$
$\inf \left\{1 - \frac{1}{r}\ \rho\left(\frac{x+y}{2}\right) : x, y \in X_\rho, \rho(x) \leq r, \rho(y) \leq r, \rho\left(\frac{x-y}{2}\right) \geq \varepsilon r\right\}.$

Existence of best approximants; example of applications

Theorem B (Kozlowski, JAT 2021)

Let $L_\rho(\Sigma)$ be a modular function space, where ρ is a convex function modular satisfying $(UUC2)$. Let $C = L_\rho(\Gamma)$ be a modular function subspace of $L_\rho(\Sigma)$, where Γ is a σ-subalgebra of Σ. Assume $x \in L_\rho(\Sigma)$ is such that $d_\rho(x, C) = \inf_{y \in C} \rho(x - y) < \infty$. There exists a unique $x_0 \in C$ such that $\rho(x - x_0) = d_\rho(x, C)$.

Since L^p and Orlicz spaces are cases of modular function spaces, this Theorem can be used for proving existence and uniqueness of the conditional expectation, the p-predictor and the φ-approximant, respectively. This Theorem may be applied to analogous approximation / prediction problems for a multitude of other spaces, e.g. variable Lebesgue spaces.

Modular projection - definitions

1. $X_\rho(K) = \{x \in X_\rho : d_\rho(x, K) < \infty\}$, where $K \subset X_\rho$.
2. Given $x \in X_\rho(K)$, any $y \in K$ such that $\rho(y - x) = d_\rho(x, K)$ is called a best ρ-approximant of x with respect to K.
3. By $P_K(x)$ we will denote the set (possibly empty) of all best ρ-approximants of x with respect to K.
4. A set-valued operator $P_K : X_\rho(K) \to 2^K$ is called a modular projection onto K.
5. We say that K is ρ-proximinal at $x \in X_\rho(K)$ if $P_K(x)$ is non-empty.
6. We say that K is ρ-proximinal if it is ρ-proximinal at every $x \in X_\rho(K)$.
7. We say that a modular projection P_K is $\rho - \zeta$ upper semicontinuous ($\rho - \zeta$ u.s.c) at $x_0 \in X_\rho(K)$ provided that for each sequence of elements $x_n \in X_\rho(K)$ with $x_n \xrightarrow{\rho} x_0$ and each $T(\zeta)$-open set V containing $P(x_0)$, $P(x_n) \subset V$ for n sufficiently large.

Modular projection - continuity results

We say that a sequence $\{y_n\}$ of elements of $y_n \in K$, is minimising for $x \in X_\rho(K)$ if $\lim_{n\to\infty} \rho(x - y_n) = dist_\rho(x, K)$. We say that $\emptyset \neq K \subset X_\rho$ is approximatively ζ-compact if for every $x \in X_\rho(K)$ and every sequence $\{y_n\}$ minimising for x, there exists a subsequence $\{y_{n_k}\}$ and an element $y \in K$ such that $y_{n_k} \xrightarrow{\zeta} y$.

Theorem C (Kozlowski, 2022)

Let (X_ρ, ρ, ζ) be a modulated ρ-complete LTI-space, where ρ is uniformly continuous on bounded sets. Let K be approximatively ζ-compact. Then

1. K is ρ-proximinal.
2. $P_K(x)$ is sequentially ζ-compact for every $x \in X_\rho(K)$
3. P_K is ρ-ζ upper semicontinuous at every $x \in X_\rho(K)$.

A modular ρ is called uniformly continuous on bounded sets if for every ρ-bounded set C and every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\rho(x + y) - \rho(x)| < \varepsilon$, whenever $x \in C$, $y \in X_\rho$, and $\rho(y) < \delta$.

Modular projection - continuity results - references

1. Reich, JMAA 1978, Prop. 2.1 (for continuous seminorms in Hausdorff topological vector spaces);
2. Deutsch, JAT 1980, Theorem 2.7 (in normed spaces with convergence);
3. Geobel and Reich, "Uniform convexity, hyperbolic geometry and nonexpansive mappings", 1984, Prop. 2.7: Projections in uniformly convex Banach spaces are continuous;

Theorem D (Kozlowski, JAT 2021)

Let (X_ρ, ρ, ζ) be a modulated ρ-complete LTI-space, where ρ is uniformly convex satisfying the Δ_2-type condition. Let $K \subset X_\rho(K)$ be nonempty, convex, ρ-bounded and ζ-closed. Then the modular projection is a single-value operator (by Theorem A) and is ρ-ζ upper semicontinuous at every $x \in X_\rho(K)$.

Application to variable Lebesgue spaces

Variable Lebesgue space $L^{p(\cdot)}$ as the modular space (actually a modular function space) is defined by the modular $\rho(x) = \int_{[0,1]} |x(t)|^{p(t)} dm(t)$, where $1 \leq p_0 \leq p(t) \leq p_1 < +\infty$ for every $t \in [0,1]$. Let us denote by $\mathcal{P}_i$ set of all polynomials of degree less than or equal to i. Define $\mathcal{R}_m^k = \left\{ \frac{g}{h} : g \in \mathcal{P}_k, h \in \mathcal{P}_m, h > 0 \, in \, [0,1] \right\}$.

Theorem E (Kozlowski, 2022)

Under these assumptions, $\mathcal{R}_m^k$ is ρ-proximinal, $P_K(x)$ is sequentially ζ-compact for every $x \in L^{p(\cdot)}$ and P_K is $\rho - \zeta$ upper semicontinuous, where ζ denotes convergence $a.e.$ in $[0,1]$.

Proof: We can show that $\mathcal{R}_m^k$ is ρ-boundedly $a.e.$-compact, which implies that it is approximatively $a.e.$-compact. Next, ρ in this case has the Δ_2-type property which implies that ρ is uniformly continuous on bounded sets. We can then apply Theorem C.