A class of set-valued mappings arising in mathematical economics

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A. M. Rubinov in his works "Turnpike sets in discrete disperse dynamical systems", Sib. Math. J. **21** (1980), 136–146 and "Multivalued mappings and their applications in economic mathematical problems" Nauka, Leningrad, 1980 introduced a discrete disperse dynamical system generated by a set-valued mapping acting on a compact metric space, which were studied in

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A. J. Zaslavski, Turnpike sets of continuous transformations in compact metric spaces, Sib. Math. J. **23** (1982), 136–146.

A. J. Zaslavski, Uniform convergence to global attractors for discrete disperse dynamical systems, Nonlinear Dynamics and System Theory **4** (2007), 315–325.

A. J. Zaslavski, Convergence of trajectories of discrete dispersive dynamical systems, Communications in Mathematical Analysis **4** (2008), 10–19.

A. J. Zaslavski, Structure of trajectories of discrete dispersive dynamical systems, Communications in Mathematical Analysis **6** 2009, 1–9.

This disperse dynamical system has prototype in the mathematical economics In particular, it is an abstract extension of the classical von Neumann-Gale model. Our dynamical system is described by a compact metric space of states and a transition operator which is set-valued. Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system.

Let (X, ρ) be a compact metric space and let $a : X \to 2^X \backslash \{\emptyset\}$ be a set-valued mapping whose graph

$$\text{graph}(a) = \{(x, y) \in X \times X : \; y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x) : \; x \in E\} \text{ and } a^0(E) = E.$$

By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

We study convergence and structure of trajectories of the dynamical system generated by the set-valued mapping a. This system is called a discrete dispersive dynamical system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Let $T_2 > T_1$ be integers. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \in \{T_1, \dots, T_2 - 1\}$.

Define

$$\Omega(a) = \{z \in X : \quad \text{for each } \epsilon > 0$$

$$\text{there is a trajectory } \{x_t\}_{t=0}^{\infty}$$

$$\text{such that } \liminf_{t \to \infty} \rho(z, x_t) \le \epsilon\}.$$

Clearly, $\Omega(a)$ is a nonempty closed subset of (X, ρ). In the literature the set $\Omega(a)$ is called a global attractor of a. Note that in the works by A. M. Rubinov $\Omega(a)$ is called a turnpike set of a. This terminology is motivated by mathematical economics.

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y) : \ y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have

$$\lim_{t \to \infty} \rho(x_t, \Omega(a)) = 0.$$

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t \to \infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) \subset B$.

Let $\phi : X \to R^1$ be a continuous function such that

$$\phi(z) \geq 0 \text{ for all } z \in X,$$

$$\phi(y) \leq \phi(x) \text{ for all } x \in X \text{ and all } y \in a(x).$$

It is clear that the function ϕ is a Lyapunov function for the dynamical system generated by the mapping a.

It should be mentioned that in mathematical economics usually X is a subset of the finite-dimensional Euclidean space and ϕ is a linear functional on this space. Our goal is to study approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a program satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given.

The following theorem theorem was obtained in A. J. Zaslavski, Structure of trajectories of discrete dispersive dynamical systems, Communications in Mathematical Analysis **6** 2009, 1–9.

Theorem 1 *The following properties are equivalent:*

(1) If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ *satisfies* $x_{t+1} \in a(x_t)$ *and* $\phi(x_{t+1}) = \phi(x_t)$ *for all integers* t*, then*

$$\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a).$$

(2) For each $\epsilon > 0$ *there exists a natural number* $T(\epsilon)$ *such that for each trajectory* $\{x_t\}_{t=0}^{\infty} \subset X$ *satisfying* $\phi(x_t) = \phi(x_{t+1})$ *for all integers* $t \geq 0$ *the inequality* $\rho(x_t, \Omega(a)) \leq \epsilon$ *holds for all integers* $t \geq T(\epsilon)$*.*

For each $x \in X$ set

$$\pi(x) = \sup\{\lim_{t\to\infty} \phi(x_t) :$$

$$\{x_t\}_{t=0}^{\infty} \text{ is a trajectory } \text{ and } x_0 = x\}.$$

The function π plays an important role in our study.

The following two useful results were also obtained in the same paper.

Prop 1 *Let $x \in X$. Then there is a trajectory $\{x_t\}_{t=0}^{\infty}$ such that $x_0 = x$ and $\pi(x) = \lim_{t\to\infty} \phi(x_t)$.*

Prop 2 *The function $\pi : X \to R^1$ is upper semicontinuous.*

It is clear that for each $x \in X$ and each $y \in a(x)$

$$\pi(y) \leq \pi(x),$$

for each $x \in X$

$$\pi(x) \leq \phi(x)$$

and that for each $x \in X$ and each natural number n

$$\pi(x) \leq \sup\{\phi(y) : y \in a^n(x)\}.$$

It is easy to see that the following proposition holds.

Prop 3 *Let $x \in X$ and $\{x_t\}_{t=0}^{\infty} \subset X$ be a trajectory such that $x_0 = x$. Then*

$$\lim_{t \to \infty} \phi(x_t) = \pi(x)$$

if and only if for each integer $t \geq 0$

$$\pi(x_{t+1}) = \max\{\pi(z) : z \in a(x_t)\}.$$

The following useful result was also proved in the same paper.

Prop 4 *Let $x \in X$. Then*

$$\pi(x) = \lim_{n\to\infty} \sup\{\phi(y) : \in a^n(x)\}.$$

The following theorem is the first turnpike result obtained in AZ 2009. It describes the structure of optimal (with respect to the functional ϕ) trajectories of a.

Theorem 2 *Assume that property (1) of Theorem 1 holds. Let* $\epsilon > 0$ *and* $x \in X$. *Then there exist* $\delta > 0$ *and a natural number* L *such that for each integer* $T > 2L$ *and each trajectory* $\{x_t\}_{t=0}^T$ *satisfying*

$$x_0 = x \text{ and } \phi(x_T) \geq \pi(x_0) - \delta$$

the following inequality holds:

$$\rho(x_t, \Omega(a)) \leq \epsilon,\ t = L, \dots, T - L.$$

We use the following property introduced in AZ 2009.

(P) If $x_1, x_2 \in \Omega(a)$ and $\phi(x_1) = \phi(x_2)$, then $x_1 = x_2$.

Note that the property (P) holds for many models of economic dynamics for which $\Omega(a)$ is a subinterval of a line.

The next convergence result was established in AZ 2009.

Theorem 3 *Assume that property (P) holds. Then each trajectory of a converges to an element of $\Omega(a)$.*

It is not difficult to see that the following result holds.

Prop 5 *Assume that property (P) holds and that* $\{x_t\}_{t=0}^{\infty}$ *is a trajectory of* a *such that*

$$\lim_{t \to \infty} \phi(x_t) = \pi(x).$$

Then by Theorem 7 there exists

$$F(x) = \lim_{t \to \infty} x_t,$$

the equality

$$\phi(F(x)) = \lim_{t \to \infty} \phi(x_t) = \pi(x)$$

holds and moreover, $F(x)$ *is a unique element of* $\Omega(a)$ *belonging to* $\phi^{-1}(\pi(x))$.

In the sequel if property (P) holds, then for each $x \in X$ we denote by $F(x)$ the unique element of

$$\Omega(a) \cap \phi^{-1}(\pi(x)).$$

The following theorem is the second turnpike result obtained in AZ 2009. It describes the structure of optimal (with respect to the functional ϕ) trajectories of a.

Theorem 4 *Assume that property (P) and property (1) of Theorem 1 hold. Let $\epsilon > 0$ and $x \in X$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$ and each trajectory $\{x_t\}_{t=0}^{T}$ satisfying*

$$x_0 = x \text{ and } \phi(x_T) \geq \pi(x) - \delta$$

the following inequality holds:

$$\rho(x_t, F(x)) \leq \epsilon,\ t = L, \ldots, T - L.$$

Theorems 6 and 9 establish the turnpike properties for approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a program satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given. In Theorem 6, the turnpike is the set $\Omega(a)$ while in Theorem 9, the turnpike is a point $F(x)$.

We obtain generalizations of these results which show that the turnpike properties still hold in the case when x_0 is not necessarily x but a point close to x. We also establish strong versions of the turnpike when $x \in \Omega(a)$. In this case for a program $\{x_t\}_{t=0}^{T}$ satisfying the assumptions of Theorem 6,

$$\rho(x_t, \Omega(a)) \leq \epsilon,\ t = 0, \ldots, T - L.$$

If it is satisfies the assumptions of Theorem 9, then

$$\rho(x_t, F(x)) \leq \epsilon,\ t = 0, \ldots, T - L.$$

Theorem 5 *Assume that property (1) of Theorem 1 holds, $x \in X$, the family of mappings $\{a^n : n = 1, 2, \ldots\}$ is equicontinuous at the point x and that $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$ and each trajectory $\{x_t\}_{t=0}^{T}$ satisfying*

$$\rho(x_0, x) \le \delta \text{ and } \phi(x_T) \ge \pi(x_0) - \delta$$

the following inequality holds:

$$\rho(x_t, \Omega(a)) \le \epsilon, \ t = L, \ldots, T - L.$$

Theorem 6 *Assume that the property (P) and property (1) of Theorem 1 hold, $x \in X$, the family of mappings $\{a^n : n = 1, 2, \dots\}$ is equicontinuous at the point x and that $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$ and each trajectory $\{x_t\}_{t=0}^T$ satisfying*

$$\rho(x_0, x) \leq \delta \text{ and } \phi(x_T) \geq \pi(x_0) - \delta$$

the following inequality holds:

$$\rho(x_t, F(x)) \leq \epsilon,\ t = L_, \dots, T - L.$$

Theorem 7 *Assume that the property (1) of Theorem 1 holds,* $x \in \Omega(a)$*, the family of mappings* $\{a^n : n = 1, 2, \dots\}$ *is equicontinuous at the point* x *and that* $\epsilon > 0$*. Then there exist* $\delta > 0$ *and a natural number* L *such that for each integer* $T > L$ *and each trajectory* $\{x_t\}_{t=0}^{T}$ *satisfying*

$$\rho(x_0, x) \leq \delta \text{ and } \phi(x_T) \geq \pi(x_0) - \delta$$

the following inequality holds:

$$\rho(x_t, \Omega(a)) \leq \epsilon, \ t = 0, \dots, T - L.$$

Theorem 8 *Assume that property (P) and property (1) of Theorem 1 hold, $x \in \Omega(a)$, the family of mappings $\{a^n : n = 1, 2, \dots\}$ is equicontinuous at the point x and that $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > L$ and each trajectory $\{x_t\}_{t=0}^T$ satisfying*

$$\rho(x_0, x) \le \delta \text{ and } \phi(x_T) \ge \pi(x_0) - \delta$$

the following inequality holds:

$$\rho(x_t, F(x)) \le \epsilon, \ t = 0, \dots, T - L.$$

Theorem 9 *Let property (1) of Theorem 1 hold and let ϵ be a positive number. Then there exists an integer $L \geq 1$ such that for every natural number $T > L$ and every trajectory $\{x_t\}_{t=0}^T$ the inequality*

$$\mathit{Card}(\{t \in \{0, \dots, T\} : \ \rho(x_t, \Omega(a)) > \epsilon\}) \leq L$$

is valid.

Theorem 10 *Assume that the function π is continuous at any point of $\Omega(a)$. Let $\epsilon \in (0,1)$. Then there exist $\delta > 0$ and a natural number L such that for every natural number $T > 2L$ and every trajectory $\{x_t\}_{t=0}^{T}$ satisfying*

$$\phi(x_T) \geq \pi(x_0) - \delta$$

there exists an integer $\tau \in \{0, \dots, L\}$ such that

$$\rho(x_t, \Omega(a)) \leq \epsilon, \ t = \tau, \dots, T - L.$$

Moreover, if $\rho(x_0, \Omega(a)) \leq \delta$, then $\tau = 0$.

Our dynamical system has a prototype in the economic growth theory. We mean the following von Neumann-Gale model. Consider the Euclidean space R^n equipped with the inner product $\langle \cdot, \cdot \rangle$ which induces the Euclidean norm $\| \cdot \|$, $\rho(x, y) = \|x - y\|$, $x, y \in R^n$. Let $R^n_+ = \{x = (x_1, \ldots, x_n) : \ x_i \geq 0, \ i = 1, \ldots, n\}$, $p = (p_1, \ldots, p_n) \in R^n$, $p_i > 0$, $i = 1, \ldots, n$ and $X = \{x \in R^n_+ : \ \langle p, x \rangle \leq 1\}$. The model is determined by a set-valued $a : X \to 2^X \setminus \emptyset$ such that its graph is a convex set and $a(\lambda x) = \lambda a(x)$ for every $x \in X$ and every $\lambda \geq 0$. Usually, it turns out that for this mapping a, $\Omega(a) = \{\lambda \widehat{x} : \ \lambda \in [0, 1]\}$, where $\widehat{x} \in X$ satisfies $\langle p, \widehat{x} \rangle = 1$, property (1) of Theorem 1 holds and the function π is continuous at any point of $\Omega(a)$.

We obtain a turnpike result for approximate solutions of problems

$$\psi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a trajectory satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given, and the function ψ satisfies certain assumptions but it is not necessarily the Lyapunov function ϕ.

We assume that the following assumption holds.

(A1) For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in X$ satisfying $\rho(x, \Omega(a)) \leq \delta$ there exist $\xi_1, \xi_2 \in \Omega(a)$ such that

$$\rho(x, \xi_i) \leq \epsilon,\ i = 1, 2 \text{ and } a(\xi_1) \subset a(x) \subset a(\xi_2).$$

Let $\mathfrak{M}$ be a set of functions $\psi : X \to [0, \infty)$ such that $\phi \in \mathfrak{M}$ and the following assumption holds.

(A2) For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $\xi_1, \xi_2 \in \Omega(a)$ satisfying $\phi(\xi_1) \leq \phi(\xi_2) - \epsilon$, each integer $n \geq 1$ and each $\psi \in \mathfrak{M}$,

$$\sup\{\psi(z) : \ z \in a^n(\xi_1)\} + \delta \leq \sup\{\psi(z) : \ z \in a^n(\xi_2)\}.$$

(Note that (A2) holds if $\mathfrak{M} = \{\phi\}$.)

Theorem 11 *Assume that the function* π *is continuous at any point of* $\Omega(a)$. *Let* $\epsilon > 0$. *Then there exist* $\delta > 0$ *and a natural number* L *such that for every natural number* $T > 5L$, *every* $\psi \in \mathfrak{M}$ *and every trajectory* $\{x_t\}_{t=0}^{T}$ *satisfying*

$$\psi(x_T) \geq \sup\{\psi(z) : \ z \in a^n(x_0)\} - \delta$$

there exists an integer $\tau \in \{0, \ldots, 2L\}$ *such that*

$$\rho(x_t, \Omega(a)) \leq \epsilon, \ t = \tau, \ldots, T - L.$$

Moreover, if $\rho(x_0, \Omega(a)) \leq \delta$, *then* $\tau = 0$.