A new approach to Williams' problem in symbolic dynamics

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In symbolic systems, one studies shift invariant subspaces of $\{0, 1\}^{\mathbb{Z}}$ that are defined by finitely many forbidden words. Together with their left-shift, such systems comprise one of the most fundamental dynamical systems in Mathematics, called subshifts of finite type (SFTs).

Conjugacy and eventual conjugacy of SFTs can be expressed in terms of equivalence relations between adjacency matrices. For instance, eventual conjugacy of edge-shifts defined by adjacency matrices A and B is equivalent to the following definition.

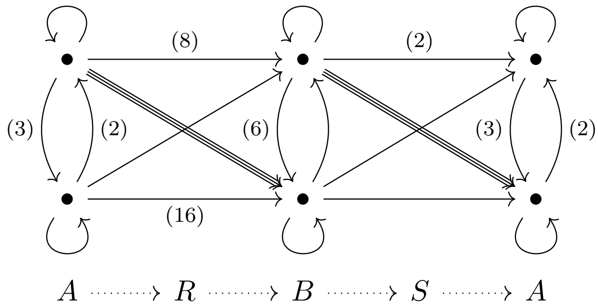
Definition 1. *Let A and B be square matrices with entries in $\mathbb{N} \cup \{0\}$. We say that A and B are* shift equivalent *of lag $1 \leq m \in \mathbb{N}$ if there are matrices R and S with entries in $\mathbb{N} \cup \{0\}$ such that*

$$AR = RB, \;\; BS = SA, \;\; A^m = RS, \;\; B^m = SR. \tag{1}$$

Example 1. *Take the matrices*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \;\; B = \begin{bmatrix} 1 & 6 \\ 1 & 1 \end{bmatrix}, \;\; R = \begin{bmatrix} 8 & 3 \\ 1 & 16 \end{bmatrix}, \;\; S = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

Then A and B are shift equivalent via R and S with lag 3, and one may think of the relations in equation (1) as ways of identifying paths between two vertices in the following figure:



A result of Williams states that *shift equivalence coincides with eventual conjugacy* of the associated SFTs, and asks if SFTs which are eventually conjugate must in fact be conjugate. From deep results we know that the answer is negative, but concrete counterexamples are difficult produce.

A recent result of Carlsen, Eilers and myself states that an additional compatibility condition on identifications between paths coming from equation (1) gives an equivalent formulation of conjugacy of SFT. This new formulation lends itself available to computer experimentation, as well as the development of the theory, in the hope of finding new counterexamples to Williams' problem.

Goals of project

- Learn about SFTs, their conjugacy, eventual conjugacy, and equivalent matrix formulations.
- Try to orient a new relation between SFTs in-between conjugacy and eventual conjugacy.
- Test new relations between SFTs, even for Example 1, using computer experimentation.

Prerequisites and further reading

- Students should have mastery in first year linear algebra, basic graph theory, and in some programming language (Python, C++, Matlab, etc...). Courses in topology and dynamics will be useful for understanding the theoretical background, but are not mandatory.
- I recommend to read Chapters 2, 6 and 7 of “An Introduction to Symbolic Dynamics and Coding” by Lind and Marcus.