Non-commutative peak points in operator algebras

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This talk is based on work from several projects, some of which joint with Ian Thompson.

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Theorem (Choquet, 1956)

Every point $x \in X$ admits a representing measure concentrated on the Choquet boundary of $\mathcal{F}$. In particular, the Choquet boundary is a boundary for $\mathcal{F}$.

The connection between the Choquet and Shilov boundaries

$x \in X$ is a peak point for $\mathcal{F}$ if there is $f \in \mathcal{F}$ with $\|f\| = 1$ such that $f(x) = 1$ and

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Conclusion: the Shilov boundary of $\mathcal{F}$ is the closure of the Choquet boundary.

Operator algebras and C*-envelopes

$\mathcal{A}$ unital operator algebra

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Yes – this is called the C-envelope.*

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How do we identify the C*-envelope $\mathrm{C}^*_e(\mathcal{A})$?

Non-commutative uniform algebra theory

(Recall: given a uniform algebra $\mathcal{F} \subset \mathrm{C}(X)$, a point $x \in X$ is in the Choquet boundary if there is a unique unital positive extension to $\mathrm{C}(X)$ of the map $f \mapsto f(x)$ on $\mathcal{F}$.)

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Let $\mathcal{J} \subset \mathrm{C}^*(\mathcal{A})$ denote the intersection of the kernels of all boundary representation for $\mathcal{A}$. Then, the C^*-envelope of $\mathcal{A}$ is $\mathrm{C}^*(\mathcal{A})/\mathcal{J}$.

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Motivating question for this talk

What about non-commutative peak points?

Going farther: what are non-commutative points?

(Recall: given a uniform algebra $\mathcal{F} \subset \mathrm{C}(X)$, a point $x \in X$ is a peak point for $\mathcal{F}$ if there is $f \in \mathcal{F}$ with $\|f\| = 1$ such that $\{x\} = \{z \in X : |f(z)| = 1\}$.)

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All three interpretations are reasonable, and offer advantages over the others.

Peak projections

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Akemann's non-commutative topology: a projection $q \in \mathrm{C}^*(\mathcal{A})^{**}$ is closed if it is the weak-$*$ limit of a decreasing net of contractions in $\mathrm{C}^*(\mathcal{A})$.

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Definition (Hay, 2007)

A projection q is a peak projection for $\mathcal{A}$ if there is a contraction $a \in \mathcal{A}$ such that $aq = q$ and $\|ap\| < 1$ for every closed projection $p \in \mathrm{C}^*(\mathcal{A})^{**}$ orthogonal to q.

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Theorem (Glicksberg, 1962)

Let $\mathcal{F}$ be a uniform algebra on a compact metric space X. A closed set $E \subset X$ is a peak set for $\mathcal{F}$ if and only if $\chi_E \in \mathcal{F}^{\perp\perp}$.

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Theorem (Blecher, Hay, Neal, Read, 2007–2012)

Assume that $\mathcal{A}$ is separable. Then, a closed projection $q \in \mathrm{C}^*(\mathcal{A})^{**}$ is a peak projection for $\mathcal{A}$ if and only if $q \in \mathcal{A}^{\perp\perp}$.

Peak representations

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We say that π is a

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If π is a peak representation, then $\ker \sigma \subset \ker \pi$ implies that $\sigma \cong \pi$.
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Definition

We say that π is a local peak representation for $\mathcal{A}$ if there is $A \in \mathbb{M}_n(\mathcal{A})$ with the property that

$$\|\pi^{(n)}(A)\| > \|P_F \sigma^{(n)}(A)|_F\|$$

for every $\sigma \in \mathcal{U}_\pi$ and F finite-dimensional subspace.

Application: residually finite-dimensional C*-envelopes

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Peak states

$\mathcal{A} \subset B(\mathcal{H})$ unital operator algebra

Definition

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None of these implications can be reversed.

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Let $\mathcal{A} \subset B(\mathcal{H})$ be a unital operator algebra that has the factorization property inside of $\mathrm{C}^*(\mathcal{A})$. Then, every pure state of $\mathrm{C}^*(\mathcal{A})$ is a peak state for $\mathcal{A}$, and the support projection of every character of $\mathrm{C}^*(\mathcal{A})$ is a peak projection for $\mathcal{A}$.

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Examples:

- $\mathcal{T}_n \subset \mathbb{M}_n$ (Cholesky)
- $H^\infty(\mathbb{D}) \subset L^\infty(\mathbb{T}, m)$
- Finite maximal subdiagonal algebras (Arveson, 1967)
- some nest algebras (Power, 1986)

Application: the hyperrigidity conjecture

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Arveson's hyperrigidity conjecture (2011)

Assume that every **irreducible** $*$-representation of $C^*(\mathcal{A})$ is a boundary representation for $\mathcal{A}$. Then, every $*$-representation enjoys the unique extension property with respect to $\mathcal{A}$.

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X be a compact metric space, $\mathcal{F} \subset C(X)$ uniform algebra
$\pi : C(X) \to B(\mathfrak{H})$ unital $*$-representation
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Arveson's local hyperrigidity theorem (2011)

Let $x \in X$ be a point in the Choquet boundary of $\mathcal{F}$. Then,

$$\lim_{\delta \to 0} \|(\pi(f) - \Pi(f))E_\pi(B(x,\delta))\| = 0, \quad f \in C(X).$$

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Thank you!