Self-similar groupoid actions and C^*-algebras

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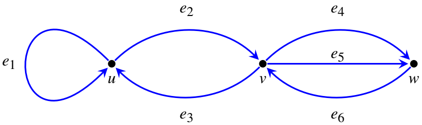
Outline

- We recall the concept of a self-similar groupoid action (G, E) on the path space of a finite graph with examples.
- We describe the structure of the associated C^*-algebra $C^*(G, E)$ and mention some properties of the Exel-Pardo étale groupoid $\mathcal{G}(G, E)$.
- We review some facts about skew products and semi-direct products of groupoids, and prove a kind of Takai duality.
- We indicate how to compute the Crainic-Moerdijk homology of $\mathcal{G}(G, E)$ in some cases, and compare it with the K-theory of $C^*(G, E)$.
- We introduce the Higman-Thompson group associated to (G, E) using G-tables and relate it to the topological full group of $\mathcal{G}(G, E)$, which is isomorphic to a subgroup of unitaries in the algebra $C^*(G, E)$.

Graphs and trees

- Let $E = (E^0, E^1, r, s)$ be a finite directed graph with no sources.

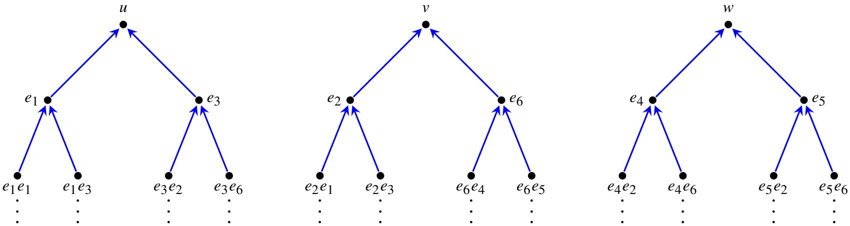


- The set of paths of length k is

$$E^k = \{e_1 e_2 \cdots e_k : e_i \in E^1,\ r(e_{i+1}) = s(e_i)\}.$$

- The space of finite paths is $E^* := \bigcup_{k \geq 0} E^k$ and E^∞ is the infinite path space with the topology given by $Z(\alpha) = \{\alpha\xi : \xi \in E^\infty\}$ for $\alpha \in E^*$.
- The set E^* is indexing the vertices of a forest T_E, where the level n has $|E^n|$ vertices.

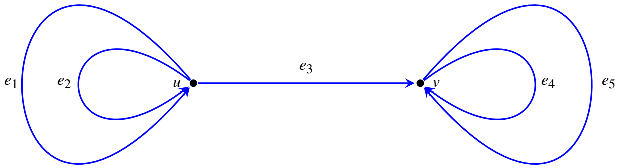
- The forest T_E looks like



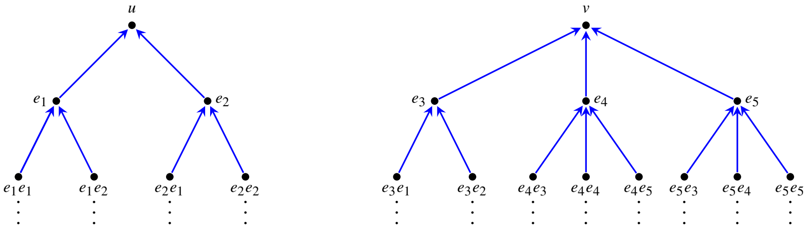
- A partial isomorphism of T_E is given by a bijection $g : u_1E^* \to u_2E^*$ preserving length and such that $g \cdot (\alpha e) \in (g \cdot \alpha)E^1$ for $\alpha \in E^k$ and $e \in E^1$.
- The set $\mathrm{PIso}(T_E)$ forms a discrete groupoid with unit space E^0.
- In this example, $\mathrm{PIso}(T_E)$ is transitive, but it could happen that there is no bijection $g : u_1E^* \to u_2E^*$ for $u_1 \neq u_2$.
- $\mathrm{PIso}(T_E)$ could be a group bundle. If $|E^0| = 1$, then $\mathrm{PIso}(T_E) = \mathrm{Aut}(T_E)$ is a group.

Graphs and trees

- Let E be the graph



with forest T_E



Obviously, the trees uE^* and vE^* are not isomorphic and $\mathrm{PIso}(T_E)$ is a group bundle.

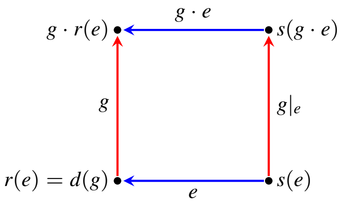
Self-similar groupoid actions

- Let E be a finite directed graph with no sources, and let G be a groupoid with unit space E^0. We denote by d and t the domain and target maps.
- **Definition**. A *self-similar action* (G, E) on the path space of E is given by a faithful groupoid homomorphism $G \to \mathrm{PIso}(T_E)$ such that for every $g \in G$ and every $e \in d(g)E^1$ there exists a unique $h \in G$ denoted by $g|_e$ and called the restriction of g to e such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \text{ for all } \mu \in s(e)E^*.$$

- We have

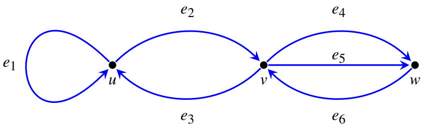
$$d(g|_e) = s(e),\ t(g|_e) = s(g \cdot e) = g|_e \cdot s(e),\ r(g \cdot e) = g \cdot r(e).$$



In general $s(g \cdot e) \neq g \cdot s(e)$, i.e. the source map is not G-equivariant.

- A self-similar action (G, E) is said to be level transitive if it is transitive on each E^n. The action is level transitive iff its extension to $\partial T_E = E^\infty$ is minimal.

- **Example 1**. Let E be the graph



- Consider the groupoid G with generators a, b, c and define the self-similar action (G, E) given by

$$a \cdot e_1 = e_2,\ \ a|_{e_1} = u,\ \ a \cdot e_3 = e_6,\ \ a|_{e_3} = b,$$

$$b \cdot e_2 = e_5,\ \ b|_{e_2} = a,\ \ b \cdot e_6 = e_4,\ \ b|_{e_6} = c,$$

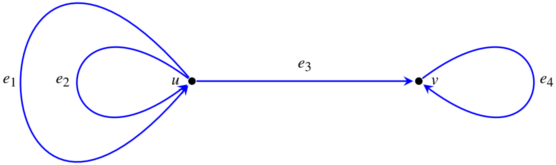
$$c \cdot e_4 = e_2,\ \ c|_{e_4} = a^{-1},\ \ c \cdot e_5 = e_6,\ \ c|_{e_5} = b.$$

- Then for example

$$b \cdot e_2 e_1 = e_5(b|_{e_2} \cdot e_1) = e_5(a \cdot e_1) = e_5 e_2.$$

- Note that the action of G is level transitive. It can be shown that G is a transitive groupoid with isotropy $\mathbb{Z}$.

- **Example 2**. Let E be the graph



- Let $G = \langle a, b, c \rangle$ with

$$a \cdot e_1 = e_2,\ a|_{e_1} = a,\ a \cdot e_2 = e_1,\ a|_{e_2} = a,$$
$$b \cdot e_1 = e_1,\ b|_{e_1} = a,\ b \cdot e_2 = e_2,\ b|_{e_2} = b,$$
$$c \cdot e_1 = e_3,\ c|_{e_1} = a,\ c \cdot e_2 = e_4,\ c|_{e_2} = c.$$

- The action is level transitive and $G_u^u = \langle a, b \rangle$ is isomorphic to the lamplighter group $L = \mathbb{Z}_2 \wr \mathbb{Z} \cong (\bigoplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}$.

- Given a self-similar groupoid action (G,E), the C^*-algebra $C^*(G,E)$ is the Cuntz-Pimsner algebra of the C^*-correspondence $\mathcal{M}$ over $C^*(G)$, where

$$\mathcal{M} = \mathcal{M}(G,E) = \mathcal{X}(E) \otimes_{C(E^0)} C^*(G).$$

- Here $\mathcal{X}(E) = C(E^1)$ is the C^*-correspondence over $C(E^0)$ associated to the graph E and $C(E^0) = C(G^{(0)}) \subseteq C^*(G)$.
- We have $\mathcal{L}(\mathcal{M}) \cong \mathcal{K}(\mathcal{M}) \cong M_n \otimes C^*(G)$, where $n = |E^1|$.
- The right action of $C^*(G)$ on $\mathcal{M}$ is the usual one and the left action is given by

$$W : G \to \mathcal{L}(\mathcal{M}), \quad W_g(i_e \otimes a) = \begin{cases} i_{g\cdot e} \otimes i_{g|_e} a & \text{if } d(g) = r(e) \\ 0 & \text{otherwise,} \end{cases}$$

where $i_e \in C(E^1)$ and $i_g \in C_c(G)$ are point masses for $e \in E^1$, $g \in G$ and $a \in C^*(G)$.

The C^*-algebra $C^*(G,E)$

- **Theorem**. The Cuntz-Pimsner algebra $C^*(G,E)$ is generated by U_g, P_v and S_e such that
  - $g \mapsto U_g$ is a representation of G with $U_v = P_v$ for $v \in E^0$;
  - S_e are partial isometries with $S_e^* S_e = P_{s(e)}$ and $\displaystyle\sum_{r(e)=v} S_e S_e^* = P_v$;
  - $U_g S_e = \begin{cases} S_{g\cdot e} U_{g|_e} & \text{if } d(g) = r(e) \\ 0, & \text{otherwise;} \end{cases}$ $\quad U_g P_v = \begin{cases} P_{g\cdot v} U_g & \text{if } d(g) = v \\ 0, & \text{otherwise.} \end{cases}$

- In general, $C^*(G,E)$ is a kind of Zappa-Szép product or C^*-blend. It contains the graph C^*-algebra $C^*(E)$.
- There is a gauge action γ of $\mathbb{T}$ such that $\gamma_z(U_g) = U_g$, and $\gamma_z(S_e) = zS_e$ for $z \in \mathbb{T}$.
- $C^*(G,E)$ is the closed linear span of elements $S_\alpha U_g S_\beta^*$, where $\alpha, \beta \in E^*$ and $g \in G_{s(\beta)}^{s(\alpha)}$. Here $S_\alpha := S_{e_1} \cdots S_{e_k}$ for $\alpha = e_1 \cdots e_k \in E^*$.
- If $\mathcal{F}_k$ denotes the closed linear span of $S_\alpha U_g S_\beta^*$ with $\alpha, \beta \in E^k$, then $\mathcal{F}_k \cong M_{n^k} \otimes C^*(G)$.
- The core algebra $\mathcal{F}(G,E) := C^*(G,E)^{\mathbb{T}}$ is isomorphic to $\varinjlim \mathcal{F}_k$ and $C^*(G,E)$ is the crossed product of $\mathcal{F}(G,E)$ by an endomorphism.
- This allows to compute $K_*(C^*(G,E))$ in some cases.

The Exel-Pardo groupoid

- **Theorem**. If (G,E) is pseudo free ($g \cdot e = e$ and $g|_e = s(e)$ implies $g = r(e)$), then there is a locally compact Hausdorff étale (Exel-Pardo) groupoid of germs

$$\mathcal{G}(G,E) = \{[\alpha, g, \beta; \xi] : \alpha, \beta \in E^*,\ g \in G_{s(\beta)}^{s(\alpha)},\ \xi \in \beta E^\infty\}$$

  such that

$$C^*(G,E) \cong C^*(\mathcal{G}(G,E)).$$

- The unit space of $\mathcal{G}(G,E)$ is identified with E^∞ by the map $[\alpha, s(\alpha), \alpha; \xi] \mapsto \xi$.
- The topology on $\mathcal{G}(G,E)$ is generated by the compact open bisections of the form

$$Z(\alpha, g, \beta; U) = \{[\alpha, g, \beta; \xi] \in \mathcal{G}(G,E) : \xi \in U\},$$

  where U is an open compact subset of $Z(\beta) = \beta E^\infty$.
- If G is amenable, then $C^*(G,E)$ is nuclear and $\mathcal{G}(G,E)$ is also amenable.
- $\mathcal{G}(G,E)$ is minimal iff E is G-transitive.
- $\mathcal{G}(G,E)$ is effective (essentially principal) iff
  (a) every G-circuit (a pair (g,α) with $s(\alpha) = g \cdot r(\alpha)$) has an entry;
  (b) for every $g \in G \setminus G^{(0)}$ there is $\zeta \in Z(d(g))$ such that $g \cdot \zeta \neq \zeta$.
- If $\mathcal{G}(G,E)$ is effective and minimal, then $\mathcal{G}(G,E)$ is purely infinite since it contains the graph groupoid $\mathcal{G}_E$.

Groupoid actions

- A groupoid G acts (on the right) on another groupoid H if there are a continuous open surjection $p : H \to G^{(0)}$ and a continuous map $H * G \to H$, write $(h, g) \mapsto h \cdot g$ where

$$H * G = \{(h, g) \in H \times G \mid t(g) = p(h)\}$$

  such that
- i) $p(h \cdot g) = d(g)$ for all $(h, g) \in H * G$,
- ii) $(h, g_1) \in H * G$ and $(g_1, g_2) \in G^{(2)}$ implies that $(h, g_1 g_2) \in H * G$ and

$$h \cdot (g_1 g_2) = (h \cdot g_1) \cdot g_2,$$

- iii) $(h_1, h_2) \in H^{(2)}$ and $(h_1 h_2, g) \in H * G$ implies $(h_1, g), (h_2, g) \in H * G$ and

$$(h_1 h_2) \cdot g = (h_1 \cdot g)(h_2 \cdot g),$$

- iv) $h \cdot p(h) = h$ for all $h \in H$.

Semi-direct products and skew products

- If G acts on H, then the semi-direct product or action groupoid is

$$H \rtimes G = H * G = \{(h, g) \in H \times G \mid t(g) = p(h)\},$$

  with multiplication

$$(h, g)(h' \cdot g, g') = (hh', gg'),$$

  when $t(g') = d(g)$ and $d(h) = t(h')$.

- The unit space of $H \rtimes G$ can be identified with $H^{(0)}$, and there is a groupoid homomorphism

$$\pi : H \rtimes G \to G,\ \pi(h, g) = g$$

  with kernel $\pi^{-1}(G^{(0)}) = \{(h, p(h)) \mid h \in H\}$ isomorphic to H.

- If G, Γ are groupoids and $\rho : G \to \Gamma$ is a homomorphism, also called a cocycle, the skew product groupoid $G \times_\rho \Gamma$ is the set of pairs $(g, \gamma) \in G \times \Gamma$ such that $(\gamma, \rho(g)) \in \Gamma^{(2)}$ with multiplication

$$(g, \gamma)(g', \gamma\rho(g)) = (gg', \gamma) \text{ if } (g, g') \in G^{(2)}$$

  and inverse

$$(g, \gamma)^{-1} = (g^{-1}, \gamma\rho(g)).$$

- There is a left action $\hat{\rho}$ of Γ on $G \times_\rho \Gamma$ given by

$$\gamma' \cdot (g, \gamma) = (g, \gamma'\gamma).$$

- Two homomorphisms $\phi_1, \phi_2 : G_1 \to G_2$ are similar if there is a continuous function $\theta : G_1^{(0)} \to G_2$ such that
$$\theta(t(g))\phi_1(g) = \phi_2(g)\theta(d(g))$$
for all $g \in G_1$.
- Two groupoids G_1, G_2 are similar if there exist $\phi : G_1 \to G_2$ and $\psi : G_2 \to G_1$ such that $\psi \circ \phi$ is similar to id_{G_1} and $\phi \circ \psi$ is similar to id_{G_2}.
- Let G act on H on the right. For $\pi : H \rtimes G \to G,\ \pi(h, g) = g$ we can form the skew product $(H \rtimes G) \times_\pi G$.
- This is made of triples $(h, g, g') \in H \times G \times G$ such that $p(h) = t(g)$ and $(g', g) \in G^{(2)}$, with unit space $H^{(0)} * G$ and operations
$$(h, g, g')(h', g'', g'g) = (h(h' \cdot g^{-1}), gg'', g'),$$
$$(h, g, g')^{-1} = (h^{-1} \cdot g, g^{-1}, g'g).$$
- **Theorem**. Let G, H, Γ be étale groupoids such that G acts on H and such that $\rho : G \to \Gamma$ is a homomorphism.
- Then $(H \rtimes G) \times_\pi G$ is similar to H and $(G \times_\rho \Gamma) \rtimes \Gamma$ is similar to G.

Homology

- If $\pi : X \to Y$ is a local homeomorphism between locally compact Hausdorff spaces, then for $f \in C_c(X, \mathbb{Z})$ define
$$\pi_*(f)(y) := \sum_{\pi(x)=y} f(x).$$
- It follows that $\pi_*(f) \in C_c(Y, \mathbb{Z})$.
- Given an étale groupoid G, let $G^{(1)} = G$ and for $n \geq 2$ let $G^{(n)}$ be the space of composable strings of n elements in G with the product topology.
- For $n \geq 2$ and $i = 0, ..., n$, we let $\partial_i : G^{(n)} \to G^{(n-1)}$ be the face maps defined by
$$\partial_i(g_1, g_2, ..., g_n) = \begin{cases} (g_2, g_3, ..., g_n) & \text{if } i = 0, \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & \text{if } 1 \leq i \leq n-1, \\ (g_1, g_2, ..., g_{n-1}) & \text{if } i = n. \end{cases}$$
- We define the homomorphisms $\delta_n : C_c(G^{(n)}, \mathbb{Z}) \to C_c(G^{(n-1)}, \mathbb{Z})$ given by
$$\delta_1 = d_* - t_*, \quad \delta_n = \sum_{i=0}^{n} (-1)^i \partial_{i*} \text{ for } n \geq 2.$$

Homology

- It can be verified that $\delta_n \circ \delta_{n+1} = 0$ for all $n \geq 1$.
- The Moerdijk-Crainic homology groups $H_n(G) = H_n(G, \mathbb{Z})$ are by definition the homology groups of the chain complex $C_c(G^{(*)}, \mathbb{Z})$ given by

$$0 \xleftarrow{\delta_0} C_c(G^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(G^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(G^{(2)}, \mathbb{Z}) \longleftarrow \cdots,$$

  i.e. $H_n(G) = \ker \delta_n / \mathrm{im}\, \delta_{n+1}$, where $\delta_0 = 0$.
- **Example**. For the action groupoid $\Gamma \ltimes X$ associated to a countable discrete group action $\Gamma \curvearrowright X$ on a Cantor set, it follows that

$$H_n(\Gamma \ltimes X) \cong H_n(\Gamma, C(X, \mathbb{Z})).$$

- Two equivalent groupoids have the same homology.
- **Theorem**(Ortega). For G an ample Hausdorff groupoid and $\rho : G \to \mathbb{Z}$ a cocycle, we have the following long exact sequence

$$0 \longleftarrow H_0(G) \longleftarrow H_0(G \times_\rho \mathbb{Z}) \xleftarrow{id-\rho_*} H_0(G \times_\rho \mathbb{Z}) \longleftarrow H_1(G) \longleftarrow \cdots$$

$$\cdots \longleftarrow H_n(G) \longleftarrow H_n(G \times_\rho \mathbb{Z}) \xleftarrow{id-\rho_*} H_n(G \times_\rho \mathbb{Z}) \longleftarrow H_{n+1}(G) \longleftarrow \cdots$$

  where ρ_* is the map induced by the action $\hat{\rho} : \mathbb{Z} \curvearrowright G \times_\rho \mathbb{Z}$.

Homology of $\mathcal{G}(G, E)$

- Given a self-similar action (G, E), there is a cocycle
$$\rho : \mathcal{G}(G, E) \to \mathbb{Z}, \quad \rho([\alpha, g, \beta; \xi]) = |\alpha| - |\beta|$$
with kernel
$$\mathcal{N}(G, E) = \bigcup_{k \geq 1} \mathcal{N}_k(G, E), \text{ where}$$
$$\mathcal{N}_k(G, E) = \{[\alpha, g, \beta; \xi] \in \mathcal{G}(G, E) : |\alpha| = |\beta| = k\} \cong (E^\infty \rtimes G) \times R_k,$$
and R_k is an equivalence relation on E^k.
- There is a homomorphism $\tau_k : \mathcal{N}_k(G, E) \to G$, $[\alpha, g, \beta; \xi] \mapsto g$ with kernel $E^\infty \times R_k$.
- Since $\mathcal{N}(G, E)$ is equivalent to the skew product $\mathcal{G}(G, E) \times_\rho \mathbb{Z}$, we have an exact sequence
$$0 \longleftarrow H_0(\mathcal{G}(G, E)) \longleftarrow H_0(\mathcal{N}(G, E)) \overset{id - \rho_*}{\longleftarrow} H_0(\mathcal{N}(G, E)) \longleftarrow H_1(\mathcal{G}(G, E))$$
$$\uparrow$$
$$\cdots \longrightarrow H_2(\mathcal{N}(G, E)) \longrightarrow H_2(\mathcal{G}(G, E)) \longrightarrow H_1(\mathcal{N}(G, E)) \overset{id - \rho_*}{\longrightarrow} H_1(\mathcal{N}(G, E))$$
where ρ_* is the map induced by the action $\hat{\rho} : \mathbb{Z} \curvearrowright \mathcal{G}(G, E) \times_\rho \mathbb{Z}$ which takes (γ, n) into $(\gamma, n + 1)$.
- This allows to compute $H_*(\mathcal{G}(G, E))$ in some cases and to compare it with $K_*(C^*(G, E))$.

Example

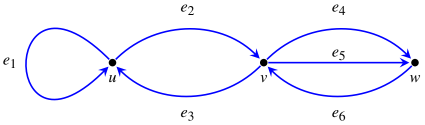
- Let G with unit space $G^{(0)} = \{u, v, w\}$ and generators a, b, c such that

$$a \cdot e_1 = e_2,\ \ a|_{e_1} = u,\ \ a \cdot e_3 = e_6,\ \ a|_{e_3} = b,$$

$$b \cdot e_2 = e_5,\ \ b|_{e_2} = a,\ \ b \cdot e_6 = e_4,\ \ b|_{e_6} = c,$$

$$c \cdot e_4 = e_2,\ \ c|_{e_4} = a^{-1},\ \ c \cdot e_5 = e_6,\ \ c|_{e_5} = b.$$

- We get a self-similar action (G, E) on



- Since G is transitive with isotropy $\mathbb{Z}$, after some computations we get

$$K_*(C^*(G, E)) \cong H_*(\mathcal{G}(G, E)) \cong 0,$$

so $\mathcal{G}(G, E)$ satisfies the HK-conjecture of Matui.

- A G-table for (G, E) with $|uE^1|$ constant is a matrix of the form

$$\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ g_1 & g_2 & \cdots & g_m \\ \beta_1 & \beta_2 & \cdots & \beta_m \end{pmatrix},$$

  where $\alpha_i, \beta_i \in E^*, g_i \in G^{s(\alpha_i)}_{s(\beta_i)}$ and $E^\infty = \bigsqcup_{i=1}^m Z(\alpha_i) = \bigsqcup_{i=1}^m Z(\beta_i)$.
- A G-table τ determines a homeomorphism $\bar{\tau}$ of E^∞ taking $\beta_i\xi$ into $\alpha_i(g_i \cdot \xi)$.
- The set of all such homeomorphisms is a countable subgroup $V_E(G)$ of $\mathrm{Homeo}(E^\infty)$, called the Higman-Thompson group.
- The topological full group of an effective étale groupoid G is

$$[\![G]\!] := \{\pi_U \mid U \subseteq G \text{ full bisection}\},$$

  where $\pi_U := t|_U \circ (d|_U)^{-1}$ from $d(U) = G^{(0)}$ to $t(U) = G^{(0)}$, which is a subgroup of $\mathrm{Homeo}(G^{(0)})$.
- The AH-conjecture of Matui claims that for G effective minimal étale with $G^{(0)}$ the Cantor set, the following sequence is exact

$$H_0(G) \otimes \mathbb{Z}_2 \xrightarrow{j} [\![G]\!]_{ab} \xrightarrow{I_{ab}} H_1(G) \to 0.$$

- **Theorem**. For a self-similar action (G, E) such that $\mathcal{G}(G, E)$ is effective, we have $V_E(G) \cong [\![\mathcal{G}(G, E)]\!]$. In particular, $[\![\mathcal{G}_E]\!] \subseteq [\![\mathcal{G}(G, E)]\!]$.
- Given a G-table $\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ g_1 & g_2 & \cdots & g_m \\ \beta_1 & \beta_2 & \cdots & \beta_m \end{pmatrix}$, the correspondence

$$\tau \mapsto T = S_{\alpha_1} U_{g_1} S^*_{\beta_1} + S_{\alpha_2} U_{g_2} S^*_{\beta_2} + \cdots + S_{\alpha_m} U_{g_m} S^*_{\beta_m}$$

  defines a faithful unitary representation of the group $V_E(G)$ in the C^*-algebra $C^*(G, E)$.
- **Corollary**. Assuming $\mathcal{G}(G, E)$ amenable, we have an exact sequence

$$1 \to U(C(E^\infty)) \to N(C(E^\infty), C^*(G, E)) \to [\![\mathcal{G}(G, E)]\!] \to 1$$

  that splits.
- Matui proved that the AH-conjecture holds for $\mathcal{G}_E$.
- **Question**. Is the AH-conjecture true for $\mathcal{G}(G, E)$?

Selected references

- M. Crainic and I. Moerdijk, *A homology theory for étale groupoids*, J. Reine Angew. Math. 521 (2000), 25–46.
- V. Deaconu, *On groupoids and C^*-algebras from self-similar actions*, New York J. of Math. 27 (2021), 923–942.
- R. Exel, E. Pardo, *Self-Similar graphs: a unified treatment of Katsura and Nekrashevych algebras*, Adv. Math. 306 (2017), 1046–1129.
- C. Farsi, A. Kumjian, D. Pask and A. Sims, *Ample groupoids: equivalence, homology, and Matui's HK conjecture*, Münster J. Math. 12 (2019), no. 2, 411–451.
- M. Laca, I. Raeburn, J. Ramagge, M. F. Whittaker, *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, Adv. Math. 331 (2018), 268–325.
- K. Matsumoto, H. Matui, *Full groups of Cuntz-Krieger algebras and Higman-Thompson groups*. Groups Geom. Dyn. 11 (2017), no. 2, 499–531.
- H. Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. London Math. Soc. (3) 104 (2012), 27–56.
- H. Matui, *Topological full groups of étale groupoids*, Operator algebras and applications–the Abel Symposium 2015, 203–230, Springer 2017.
- V. Nekrashevych, *C^*-algebras and self-similar groups*, J. Reine Angew. Math. 630 (2009) 59–123.
- V. Nekrashevych, *Self-similar groups*, Math. Surveys Monogr. 117, AMS Providence 2005.

THANK YOU!