von Neumann's inequality on the disc and on the ball

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Matrices and polynomials

Let A be an $n \times n$ matrix over $\mathbb{C}$ and let

$$p(z) = a_0 + a_1 z + \ldots + a_k z^k$$

be a polynomial. Define

$$p(A) = a_0 I + a_1 A + \ldots + a_k A^k.$$

General principle in linear algebra

Understand properties of A by studying $p(A)$ for polynomials p.

e.g. eigenvalues, minimal polynomial, Jordan canonical form.

Goal

Extend this idea to operators on Hilbert space.

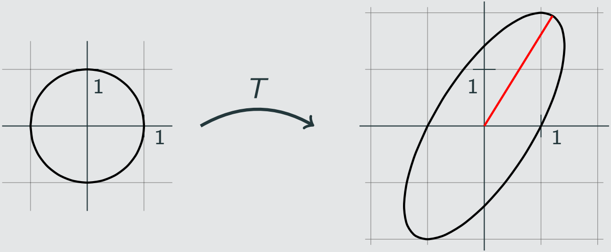
The operator norm

Let T be a linear operator on a Hilbert space $\mathcal{H}$, define

$$\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}.$$

Example

Let $\mathcal{H} = \mathbb{C}^2$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then $\|T\| = \sqrt{\sqrt{5} + 3}$.



If $\|T\| \leq 1$, we say that T is a contraction.

von Neumann's inequality

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem (von Neumann, 1951)

Let T be a contraction on a Hilbert space. Then

$$\|p(T)\| \leq \sup\{|p(z)| : z \in \overline{\mathbb{D}}\}$$

for all $p \in \mathbb{C}[z]$.

Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes.

ERHARD SCHMIDT zum 75. Geburtstag in Verehrung gewidmet.

Von JOHANN VON NEUMANN in Princeton, N. J. (USA.).

(Eingegangen am 27. 7. 1950.)

1. Einleitung.

1.1. Der Gegenstand dieser Arbeit ist die Einführung eines neuen Begriffs des *Spektrums* eines (linearen) Operators und die Herleitung seiner wesentlichsten Eigenschaften. Die Theorie gilt in allen unitären Räumen[1]), d. h. es kostet keine zusätzliche Anstrengung, sie von endlichdimensionalen [euklidischen[2])] Räumen auf unendlichdimensionale [hilbertsche[2])] Räume, separabel (abzählbar-unendlichdimensional) oder nicht, auszudehnen. Die Theorie wird daher gleich

Der obige Satz nimmt übrigens in unserer Theorie eine durchaus zentrale Stellung ein.

4.3. Wir wiederholen die zu beweisende Aussage:

Aus $|||A||| \leqq 1$ *folgt, daß* E_0 *(vgl. (12)) Spektralmenge von* A *ist.*

Oder, indem wir dies in der Definition der Spektralmenge in **3.1** substituieren und die Annahmen etwas umstellen:

Sei $f(\lambda)$ *eine rationale Funktion mit*

(13.a) $$|f(\lambda)| \leqq 1 \quad \text{für alle } \lambda \text{ mit } |\lambda| \leqq 1 .$$

Dann gilt auch:

(13.b) $f(A)$ *existiert und es ist* $|||f(A)||| \leqq 1$ *für alle* A *mit* $|||A||| \leqq 1$.

Function theory helps operator theory

Theorem (von Neumann's inequality)

If T is a contraction, then $\|p(T)\| \leq \sup_{z \in \overline{\mathbb{D}}} |p(z)|$ for all $p \in \mathbb{C}[z]$.

Can extend the map $p \mapsto p(T)$ from $\mathbb{C}[z]$ to bigger algebras of functions.

Theorem (Sz.-Nagy–Foias)

If T is a contraction, then under mild assumptions, one can make sense of $f(T)$ for bounded holomorphic functions f on $\mathbb{D}$.

General principle

Use function theory on $\mathbb{D}$ to study contractions on Hilbert space.

Applications: invariant subspace results, structure theorems for classes of contractions, ...

Operator theory helps function theory

Toy example

Consider the contraction

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If $p \in \mathbb{C}[z]$ with $p(0) = 0$, then

$$p(T) = \begin{bmatrix} 0 & p'(0) \\ 0 & 0 \end{bmatrix}.$$

By von Neumann's inequality,

$$|p'(0)| = \|p(T)\| \leq \sup\{|p(z)| : z \in \overline{\mathbb{D}}\}.$$

This is part of the Schwarz lemma.

Sz.-Nagy's dilation theorem

An operator U with $UU^* = U^*U = I$ is called unitary.

Theorem (Sz.-Nagy, 1953)

Let T be a contraction on $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary operator U on $\mathcal{K}$ such that

$$T = P_{\mathcal{H}} \quad U \big|_{\mathcal{H}}$$

We say that U is a unitary dilation of T.

Equivalently,

$$U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix}.$$

Proof of von Neumann's inequality

(1) If U is unitary, then $\sigma(U) \subset \partial\mathbb{D}$. By spectral theory,

$$\|p(U)\| = \sup\{|p(z)| : z \in \sigma(U)\} \leq \sup\{|p(z)| : z \in \overline{\mathbb{D}}\}.$$

(2) Let T be a contraction. By Sz.-Nagy's dilation theorem, T admits a unitary dilation U, so

$$p(T) = P_{\mathcal{H}} p(U)\big|_{\mathcal{H}}.$$

Thus,

$$\|p(T)\| = \|P_{\mathcal{H}} p(U)\big|_{\mathcal{H}}\| \leq \|p(U)\| \overset{(1)}{\leq} \sup\{|p(z)| : z \in \overline{\mathbb{D}}\}.$$

Multivariable theory

Multivariable theory

Goal

Extend von Neumann's inequality to operator tuples $T = (T_1, \dots, T_d)$.

1) Commutative theory: $T_i T_j = T_j T_i$.
    a) tuples of contractions: $\| T_i \| \leq 1$ for all i. Connects to polydisc $\mathbb{D}^d$.
    b) row contractions: Assume

$$\begin{bmatrix} T_1 & \cdots & T_d \end{bmatrix} : \mathcal{H}^d \to \mathcal{H}, \quad \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \mapsto \sum_{i=1}^{d} T_i x_i,$$

    is contraction. Connects to unit ball $\mathbb{B}_d = \{ z \in \mathbb{C}^d : \|z\|_2 < 1 \}$.
2) Non-commutative theory
    a) tuples of contractions.
    b) row contractions.

Commuting tuples of contractions

Theorem (Andô, 1963)

Let $T = (T_1, T_2)$ be a pair of commuting contractions. Then T dilates to a pair of commuting unitaries. Hence

$$\|p(T)\| \leq \sup\{|p(z)| : z \in \overline{\mathbb{D}}^2\} \quad \text{for all } p \in \mathbb{C}[z_1, z_2].$$

- Parrott (1970): Unitary dilation may fail for triples of commuting contractions.
- Kaijser–Varopoulos (1974), Crabb–Davie (1975): von Neumann's / Andô's inequality may fail for triples of commuting contractions.

Open question

Do there exist constants C_d such that for all commuting contractions $T = (T_1, \ldots, T_d)$ and all $p \in \mathbb{C}[z_1, \ldots, z_d]$, we have

$$\|p(T)\| \leq C_d \sup\{|p(z)| : z \in \overline{\mathbb{D}}^d\}?$$

Aside: a connection to algebraic geometry

Let $V(d,n) = \{d\text{-tuples of commuting } n \times n \text{ matrices}\} \subset \mathbb{C}^{dn^2}$.

Question

For which d, n is $V(d,n)$ irreducible?

- Motzkin, Taussky (1955): $V(2,n)$ is irreducible for all n.
- Gerstenhaber (1961), Guralnick (1992): $V(d,n)$ is reducible if $d \geq 4$ and $n \geq 4$.
- Easy: $V(d,n)$ is irreducible if $n \leq 3$.
- Holbrook, Omladič (2001): $V(3,n)$ is reducible if $n \geq 29$.
- Šivic (2012): $V(3,n)$ is irreducible if $n \leq 10$.

Open Problem

Is $V(3,n)$ irreducible for $11 \leq n \leq 28$?

A glimpse at the non-commutative theory

Theorem (Bożejko, 1989)

Let $T_1, \dots, T_d$ be not necessarily commuting contractions. Then

$$||p(T_1, \dots, T_d)|| \le \sup\{||p(U_1, \dots, U_d)|| : U_1, \dots, U_d \text{ unitary matrices}\}$$

for every polynomial p in d non-commuting variables.

Davidson–Pitts, Popescu: Dilation theory and von Neumann's inequality for non-commuting row contractions.

Muhly–Solel: Generalizations to W^*-correspondences.

Commuting row contractions

The Drury–Arveson space H^2_d is a Hilbert space of holomorphic functions on $\mathbb{B}_d$. Let $M_{z_i} : H^2_d \to H^2_d, f \mapsto z_i \cdot f$.

Then $M_z = (M_{z_1}, \dots, M_{z_d})$ is a commuting row contraction.

Theorem (Drury, Müller–Vasilescu, Arveson)

Let $T = (T_1, \dots, T_d)$ be a commuting row contraction. Then

$$\|p(T)\| \le \|p(M_z)\| = \|p\|_{\mathrm{Mult}(H^2_d)} \text{ for all } p \in \mathbb{C}[z_1, \dots, z_d].$$

Example

Let $d = 2$ and $p(z_1, z_2) = 2z_1 z_2$. Then $\|p^k\|_\infty = 1$, but $\|p^k(M_z)\| \approx \sqrt{k}$.

So if $d \ge 2$, then there does not exist a constant C_d such that for all commuting row contractions T and all $p \in \mathbb{C}[z_1, \dots, z_d]$, we have

$$\|p(T)\| \le C_d \sup\{|p(z)| : z \in \overline{\mathbb{B}_d}\}.$$

Row contractive matrices

A constant for matrices?

Question

Let $d, n \in \mathbb{N}$. Does there exist a constant $C_{d,n}$ such that for every row contraction T consisting of d commuting $n \times n$ matrices and every polynomial p, the following inequality holds:

$$\|p(T)\| \leq C_{d,n} \sup_{z \in \overline{\mathbb{B}_d}} |p(z)|.$$

Must have $C_{d,n} \xrightarrow{n \to \infty} \infty$ for all $d \geq 2$.

Spectral theory is not enough:

Example

Let $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $\sigma(T) \subset \overline{\mathbb{D}}$, but $T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

A von Neumann inequality for matrices

Theorem (H.–Richter–Shalit)

There exist constants $C_{d,n}$ such that for every row contraction T consisting of d commuting $n \times n$ matrices and every polynomial p, the following inequality holds:

$$\|p(T)\| \le C_{d,n} \sup_{z \in \overline{\mathbb{B}_d}} |p(z)|.$$

If $d \ge 2$, then the optimal constants $C_{d,n}$ satisfy

$$n^{\frac{1}{8}} \le C_{d,n} \le C(d)^{n-1},$$

where $C(d)$ is a dimensional constant related to Gleason's problem.

Key idea and applications

Key idea: reduction to interpolation problem

Suppose T is jointly diagonalizable with $\sigma(T) \subset \mathbb{B}_d$.

Given a polynomial p, find holomorphic g with

$$g\big|_{\sigma(T)} = p\big|_{\sigma(T)} \quad \text{and} \quad \|g\|_{\mathrm{Mult}(H^2_d)} \le C_{d,n} \|p\|_\infty.$$

Then

$$\|p(T)\| = \|g(T)\| \le \|g\|_{\mathrm{Mult}(H^2_d)} \le C_{d,n} \|p\|_\infty.$$

Interpolation problem is solved using multivariable Schur algorithm.

Further applications

1. Answer question about multiplier algebras on the ball (Aleman–H.–M^{c}Carthy–Richter).
2. Gleason's problem in $H^\infty(\mathbb{B}_d)$ cannot be solved contractively.

An application to nc function theory

Let $\mathfrak{CB}_d(n) = \{$strict row contractions of d commuting $n \times n$ matrices$\}$. Let $\mathfrak{CB}_d = \bigsqcup_n \mathfrak{CB}_d(n)$. An nc holomorphic function on $\mathfrak{CB}_d$ takes $\mathfrak{CB}_d(n)$ into $M_n(\mathbb{C})$ (plus axioms).

Question (Salomon–Shalit–Shamovich)

Is every levelwise uniformly continuous bounded nc holomorphic function on $\mathfrak{CB}_d$ globally uniformly continuous?

Corollary (H.–Richter–Shalit)

No.

Idea: bounded nc holomorphic on $\mathfrak{CB}_d \leftrightarrow$ multiplier of H^2_d.

Globally uniformly continuous $\leftrightarrow$ element of $\mathcal{A}_d = \overline{\mathbb{C}[z_1, \ldots, z_d]}^{\|\cdot\|_{\mathrm{Mult}(H^2_d)}}$.

von Neumann-type inequality for matrices: multipliers of H^2_d in $C(\overline{\mathbb{B}_d})$ are levelwise uniformly continuous.

Fang–Xia; Shalit: There exist multipliers of H^2_d in $C(\overline{\mathbb{B}_d})$ not in $\mathcal{A}_d$.

Summary

- von Neumann's inequality gives a fundamental link between operator theory and complex analysis.
- In the multivariable setting, several challenges arise.
- von Neumann's inequality holds for commuting row contractive matrices on the ball up to a constant.

Thank you!