Groupoids, Unitary extensions and Wavelets

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NAT, in honor of my adviser Paul S. Muhly

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Motivations: Wavelets

Definition

A **wavelet** is a vector ψ in $L^2(\mathbb{R})$ such that the family

$$\{D^j T^k \psi \ : \ j, k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$, where $T\xi(x) = \xi(x-1)$, $\xi \in L^2(\mathbb{R})$, and $D\xi(x) = \sqrt{2}\xi(2x)$.

Fact (Building wavelets via Cuntz isometries [BJ97])

Let $\sigma : \mathbb{T} \to \mathbb{T}$ be $\sigma(z) = z^2$ and $\{m_1, m_2\}$ be a filter bank: $\sum_{\sigma(w)=z} \overline{m_i(w)} m_j(w) = 2\delta_{i,j}$.
Then $\{S_1, S_2\}$, where $S_i(\xi)(z) = m_i(z)\xi(\sigma(z))$, $\xi \in L^2(\mathbb{T})$, is a Cuntz family of isometries.
***Key idea**: build the minimal unitary extension of S_1.*

The Deaconu-Renault groupoid

Fact ([Ren80, Dea95, AR97, Ren00])

- *Let X be a compact Hausdorff space and $\sigma : X \to X$ a onto local homeomorphism.*
- *The Deaconu-Renault groupoid $G(X, \sigma)$ is defined via*

$$G := G(X, \sigma) := \{(x, m-n, y) \in X \times \mathbb{Z} \times X \, : \, \sigma^m(x) = \sigma^n(y)\}$$

  endowed with the operations $(x, k, y)(y, l, z) = (x, k+l, z)$ and $(x, k, y)^{-1} = (y, -k, x)$.
- *$G(X, \sigma)$ is an étale locally compact groupoid.*

Filter banks and Cuntz isometries in $G(X, \sigma)$

Fact

- *A filter bank* $\{m_1, \ldots, m_n\}$ *is a family of functions on* X *such that* $\sum_{\sigma(y)=x} \overline{m_i(y)} m_j(y) = |\sigma^{-1}(x)| \delta_{i,j}$.
- *[IM08] A filter bank determines a Cuntz family of isometries in* $C^*(G)$.

Example ([IM08])

Let $X = \mathbb{T}$, $\sigma(z) = z^2$, and $\{m_1, m_2\}$ a filter bank. If L is the trivial unitary representation of $G(\mathbb{T}, \sigma)$, we recover the classical wavelet construction.

Imprimitivity groupoids

Fact ([MRW87])

We assume now that G is an arbitrary topological groupoid.

- *Let Z be a free and proper right G-space. We write $s : Z \to G^{(0)}$ for the moment map.*
- *Then G acts diagonally on $Z * Z$ and $G^Z := (Z * Z)/G$ is a groupoid that acts on the left on Z via*

$$[x, y] \cdot (yg) = xg.$$

- *Moreover, G^Z and G are equivalent groupoids and Z is an equivalence between them.*

Blow up groupoids

Fact

We continue to assume that G is a topological groupoid.

- *Let Y be a l.c. Hausdorff space and $\Phi : Y \to G^{(0)}$ a continuous open surjective map.*
- *Then $Z := Y * G$ is a right G-space: the moment map is $s(x, g) = s(g)$ and $(x, g)h = (x, gh)$.*

Theorem

The imprimitivity groupoid $G^Z = (Z * Z)/G$ is isomorphic to the groupoid $Y * G * Y$, where

$$Y * G * Y = \{(x, g, y) \in Y \times G \times Y \ : \ \Phi(x) = r(g) \text{ and } \Phi(y) = s(g)\},$$

is endowed with the operations $(x, g, y) \cdot (y, h, z) := (x, gh, z)$ and $(x, g, y)^{-1} := (y, g^{-1}, x)$.

Haar systems on the blow up groupoids

Fact

Assume that G is a topological groupoid endowed with a Haar system $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$ and $\Phi : Y \to G^{(0)}$ is an open continuous surjective map.

- *We begin by choosing an arbitrary full Φ-system of measures $\{\nu_u\}_{u \in G^{(0)}}$ on Y.*
- *The system of measures $\{\alpha_u\}_{u \in G^{(0)}}$ defined via*

$$\alpha_u(f) := \int_{G_u} \int_Y f(y, g) d\nu_{r(g)}(y) d\lambda_u(g), \qquad u \in G^{(0)},$$

  is a full, equivariant s-system of measures on Z.
- *It follows that the equation*

$$\beta(f)(x) := \int_{G^{\Phi(x)}} \int_Y f(x, g, y) d\nu_{s(g)}(y) d\lambda^{\Phi(x)}(g), \; x \in Y,$$

  defines a Haar system on $Y * G * Y$.

σ-systems of measures on X

Fact

- *σ is dual to the injective C^*-endomorphism $\pi : C(X) \to C(X)$ defined by $\pi(f) := f \circ \sigma$.*
- *Therefore, π has a left inverse.*

Theorem

Every left inverse of π is given by a map $\mathcal{L}_D : C(X) \to C(X)$ where D is a strictly positive, real-valued, continuous function on X such that

$$\sum_{\sigma(y)=x} D(y) = 1$$

for all $x \in X$ and $\mathcal{L}_D$ is defined by the formula

$$\mathcal{L}_D(f)(x) := \sum_{\sigma(y)=x} D(y) f(y), \qquad f \in C(X).$$

Conversely, each such $\mathcal{L}_D$ is a left inverse of π.

A sequence of blow up groupoids

Fact

*We **chose** a transfer operator given by a continuous map D.*

- *For $n \geq 0$, let $X_n = X$ and $\Phi_n : X_n \to X$ be $\Phi_n(x) = \sigma^n(x)$.*
- *Let*

$$Z_n := X_n * G = \{(z, (x, k-l, y)) \in X \times G \, : \, \sigma^n(z) = x\}.$$

- *The imprimitivity groupoid G^{Z_n} is isomorphic to the blow up groupoid $G_n := X_{\sigma^n} * G *_{\sigma^n} X$.*
- *For $n \geq 1$ define the Φ_n-system of measure $\{\nu_{n,x}\}$ on X via*

$$\nu_{n,x}\{y\} = D(\sigma^{n-1}(y)) \cdots D(y)$$

Inductive systems of C^*-correspondences

Fact

- *For $n \geq 0$, let $\mathcal{X}_n = \overline{C_c(Z_n)}^{C^*(G)}$ be the corresponding $C^*(G_n) - C^*(G)$ imprimitivity bimodule.*

Theorem

For $n \geq m \geq 0$ define $V_{n,m} : \mathcal{X}_m \to \mathcal{X}_n$ via

$$V_{n,m}(\xi)(z,(x,k-l,y)) = \xi(\sigma^{n-m}(z),(x,k-l,y))$$

for all $(z,(x,n-m,y)) \in Z_n$. Then $\{V_{n,m}\}$ is a sequence of adjointable isometries from $\mathcal{X}_m$ to $\mathcal{X}_n$ such that $V_{n,m}V_{m,k} = V_{n,k}$ for all $n \geq m \geq k \geq 0$. In particular, $\{\mathcal{X}_n, V_{n,m}\}$ is an inductive sequence of Hilbert-modules.

A "limit" groupoid

Fact

- *Let the projective system* $X_n \overset{\sigma_{n,m}}{\leftarrow} X_m$, *where* $X_n = X$ *for every* n, *and* $\sigma_{n,m} = \sigma^{m-n}$ *for all* $m \geq n$.
- *Consider the projective limit*

$$X_\infty := \{\underline{x} := (x_n)_{n\geq 1} \in X^{\mathbb{N}} \mid x_n = \sigma(x_{n+1})\}.$$

- *The map* $\sigma_\infty : X_\infty \to X_\infty$ *defined by* $\sigma_\infty(x_1, x_2, \cdots) = (\sigma(x_1), x_1, x_2, \cdots)$ *is a homeomorphism such that* $p_n \circ \sigma_\infty = \sigma \circ p_n$ *for all* n.

A "limit" groupoid, II

Fact

- *We form the right G-space* $Z_\infty := X_\infty * G = \{(\underline{x}, (x, k - l, y)) \in X \times G\}$
- *Let* $G_\infty = X_\infty * G * X_\infty$ *be the blow up groupoid.*
- *We define a full p-system* $\{\nu_x\}_{x \in X}$ *of measures on* X_∞ *via*

$$\int_{X_\infty} f_1(x_1) \cdot \dots \cdot f_n(x_n) d\nu_x(\underline{x}) = f_1(x) \Big(\sum_{\sigma(x_2)=x} D(x_2) f(x_2) \cdots \Big(\sum_{\sigma(x_n)=x_{n-1}} D(x_n) f(x_n) \Big) \cdots \Big) \Big).$$

The limit of the inductive system

Theorem

Let $\mathcal{X}_\infty = \overline{C_c(Z_\infty)}^{C^*(G)}$ *be the* $C^*(G_\infty) - C^*(G)$ *imprimitivity bimodule. Then* $\mathcal{X}_\infty$ *is isomorphic to the inductive limit* $\lim_{\to}(\mathcal{X}_n, V_{n,m})$ *in the sense of [LR07]. Indeed,* $V_{\infty,n} : C_c(Z_n) \to C_c(Z_\infty)$ *defined via*

$$V_{\infty,n}(\xi)(\underline{x}, (x_1, k-l, y)) := \xi(x_n, (x_1, k-l, y))$$

extends to an adjointable isometry for all $n \geq 0$ *such that* $V_{\infty,n} \circ V_{n,m} = V_{\infty,m}$ *for all* $n \geq m \geq 0$ *and* $\bigcup_{n\geq 0} V_{\infty,n}(\mathcal{X}_n)$ *is dense in* $\mathcal{X}_\infty$*.*

Isometries in $\mathcal{L}(\mathcal{X}_n)$

Fact

For $n \geq 0$ *and* $\xi \in C_c(Z_n)$ *define*

$$S_n\xi(z,(x,k-l,y)) = \sqrt{D(x)}\xi(\sigma(z),(\sigma(x),k-1-l,y)).$$

Theorem

S_n *is an adjointable isometry on* $\mathcal{X}_n$ *for all* $n \geq 0$ *such that* $V_{n,m}S_m = S_nV_{n,m}$ *for all* $n \geq m \geq 0$.
Moreover, for $n = 0$ *the isometry* S_0 *is given by*

$$S_D(x,k-l,y) = \begin{cases} \sqrt{D(x)} & \text{if } \sigma(x) = y,\ k-l=1 \\ 0 & \text{otherwise.} \end{cases}$$

Example

Fact

- *Let* $X = \mathbb{T}$ *and* $\sigma(z) = z^2$.
- *Let* $\{m_1, m_2\}$ *be a filter bank.*
- *Set* $D(z) = |m_1(z)|^2/2$. *Then* $\sum_{w^2=z} D(w) = 1$ *for all* $z \in \mathbb{T}$.
- *The isometry* S_0 *equals the isometry* S_{m_1} *defined earlier in the talk.*

The unitary extension of S_0

Theorem

Let $U \in \mathcal{L}(\mathcal{X}_\infty)$ be defined via

$$U\xi(\underline{x}, (x, k-l, y)) = \sqrt{D(x)}\xi(\sigma_\infty(\underline{x}), (\sigma(x), k-1-l, y))$$

for $\xi \in C_c(Z_\infty)$. The U is a unitary such that $U \circ V_{\infty,n} = V_{\infty,n} \circ S_n$ for all $n \geq 0$. In particular, U is the minimal unitary extension of $S_0 = S_D$.

Theorem

U acts as a multiplier on $C^*(G_\infty)$ via

$$(Uf)(\underline{x}, (x, k-l, y), \underline{y}) = \sqrt{D(x)}f(\sigma_\infty(\underline{x}), (\sigma(x), k-l-1, y), \underline{y})$$

for all $f \in C_c(G_\infty)$ and $(\underline{x}, (x, k-l, y), \underline{y}) \in G_\infty$.

An aplication: generalized multiresolution analysis

Theorem

Let $Y_n := V_{\infty,n}(\mathcal{X}_n)$ for all $n \geq 0$ and let $Y_n = U(Y_{n+1})$ for all $n < 0$. The sequence of submodules $\{Y_k\}$ and the unitary U form a projective multi-resolution analyses for $\mathcal{X}_\infty$. That is, $(\{Y_k\}, U)$ satisfy the following properties:

1. *Y_0 is a complemented $C^*(G)$-submodule of $\mathcal{X}_\infty$.*
2. *$Y_{n+1} = U^{-1}(Y_n)$ for all $n \in \mathbb{Z}$.*
3. *Y_n is a complemented sub-module of Y_{n+1} for all $n \in \mathbb{Z}$.*
4. *$\bigcup_{n\in\mathbb{Z}} Y_n$ is dense in $\mathcal{X}_\infty$.*

If, in addition, S_0 is a pure isometry, then $\bigcap_{n\in\mathbb{Z}} Y_n = \emptyset$.

From groupoids to Hibert spaces: unitary representations of groupoids

Definition

- A unitary representation of (G, λ) is a triple $(\mu, G^{(0)} * \mathcal{H}, L)$, where
  - μ is a quasi-invariant measure on $G^{(0)}$:

$$\int_{G^{(0)}} \int_{G_u} f(g)\Delta_\mu(g)\, d\lambda_u(g) d\mu(u) = \int_{G^{(0)}} \int_{G^u} f(g)\, d\lambda^u(g) d\mu(u).$$

  - $G^{(0)} * \mathcal{H}$ is a Hilbert bundle over $G^{(0)}$.
  - $L : G \to \text{Iso}(\mathcal{H}) = \{(r(g), L_g, s(g)) : g \in G\}$, with $L_g : \mathcal{H}(s(g)) \to \mathcal{H}(r(g))$ a Hilbert space isomorphism.
- The integrated form of a unitary representation acts on $L^2(G^{(0)} * \mathcal{H}, \mu)$ via

$$L(f)\xi(u) = \int_{G^u} f(g) L_g(\xi(s(g)))\, d\lambda^u(g).$$

Inducing unitary representations to the blow up groupoid

Fact ([Ren14])

- *Let* $(\mu, G^{(0)} * \mathcal{H}, L)$ *a unitary representation of* (G, λ) *and let* Δ_μ *the cocycle determined by* μ.
- *Let* $Z = Y * G$ *and* $\nu = \{\nu_u\}_{u \in G^{(0)}}$ *be a* Φ*-system on* Y. *The induced representation* $(m, \mathcal{K}, \operatorname{Ind} L)$ *of* $Y * G * Y$ *is defined via:*
  - *There is a measurable function* b *on* Z *such that* $b(xg)/b(x) = \Delta_\mu(g)$ *([Hol17]).*
  - *The measure* m *on* Y *is given by*

$$\int_Y f(x)\, dm(x) = \int_{G^{(0)}} \int_Y f(x)\, b(x, \Phi(x)) d\nu_u(x) d\mu(u).$$

  - *The Hilbert bundle* $\mathcal{K} = Y * \mathcal{H} = \{(x, \xi) : \xi \in \mathcal{H}(\Phi(x))\}$ *and the induced action is given by*

$$\operatorname{Ind} L_{(x,g,y)}(y, \xi) = (x, L_g \xi).$$

Example

Example

- Let $X = \mathbb{T}$ and $\sigma(z) = z^2$, and let $\{m_1, m_2\}$ be a filter bank. As before, we let D be defined by m_1.
- Let $(\mu, X * \mathcal{H}, L)$ be the trivial representation of $G(\mathbb{T}, \sigma)$: μ is the normalized Haar measure on $\mathbb{T}$; $\mathcal{H}$ is the trivial one dimensional Hilbert bundle; L is the trivial representation $L_{(x,k,y)}(y, \xi) = (x, \xi)$.
- The integrated form of L acts on $L^2(\mu)$ and $L(S_1)\xi(z) = m_1(z)\xi(z^2)$.
- The space $\mathbb{T}_\infty$ is the 2-adic solenoid and the blow up groupoid is $\mathbb{T}_\infty * G * \mathbb{T}_\infty$.
- Since μ is invariant, b is constant and we can chose it to be 1. The Hilbert bundle $\mathcal{K}$ is the trivial bundle.
- Hence the integrated form acts on $L^2(m)$ and we recover the minimal unitary extension that defined the wavelet.

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