Isomorphisms and stable isomorphisms of non-selfadjoint operator algebras

Elias Katsoulis

East Carolina University

katsoulise@ecu.edu

June 30, 2022

Isomorphisms of Operator Algebras

A C^*-dynamical system $(\mathcal{A}, \alpha)$ consists of a C^*-algebra $\mathcal{A}$ and a $*$-endomorphism

$$\alpha : \mathcal{A} \to \mathcal{A}.$$

The definition of a semicrossed product

A C^*-dynamical system $(\mathcal{A}, \alpha)$ consists of a C^*-algebra $\mathcal{A}$ and a $*$-endomorphism

$$\alpha : \mathcal{A} \to \mathcal{A}.$$

An isometric covariant representation (π, V) of $(\mathcal{A}, \alpha)$ consists of a $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an isometry $V \in B(\mathcal{H})$ so that

$$\pi(a)V = V\pi(\alpha(a)), \text{ for all } a \in \mathcal{A}.$$

Definition

The semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is the universal operator algebra associated with "all" covariant representations of $(\mathcal{A}, \alpha)$, i.e., the universal algebra generated by a faithful copy of $\mathcal{A}$ and an isometry v satisfying the covariance relations.

Definition

The semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is the universal operator algebra associated with "all" covariant representations of $(\mathcal{A}, \alpha)$, i.e., the universal algebra generated by a faithful copy of $\mathcal{A}$ and an isometry v satisfying the covariance relations.

In the case where α is an automorphism of $\mathcal{A}$, then $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is isomorphic to the subalgebra of the crossed product C^*-algebra $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ generated by $\mathcal{A}$ and the "universal" unitary u implementing the covariance relations.

The classification problem

One of the central problems in the study of semicrossed products is the classification problem, whose study spans more than 50 years. This problem asks if two semicrossed products are isomorphic as algebras exactly when the corresponding C^*-dynamical systems are outer conjugate, that is, unitarily equivalent after a conjugation.

One of the central problems in the study of semicrossed products is the classification problem, whose study spans more than 50 years. This problem asks if two semicrossed products are isomorphic as algebras exactly when the corresponding C^*-dynamical systems are outer conjugate, that is, unitarily equivalent after a conjugation.

Definition

The C^*-dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are said to be outer conjugate if there exists a $*$-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ and a unitary $u \in M(\mathcal{A})$ so that

$$\alpha(a) = u\big(\gamma^{-1} \circ \beta \circ \gamma(a)\big)\, u^*, \qquad \text{for all } a \in \mathcal{A}.$$

The classification problem first appeared in the work of Arveson (1967) and it was subsequently investigated by Peters (1984), Hadwin and Hoover (1988), Power (1992) and Davidson and Katsoulis (2008), who finally settled affirmatively the case where $\mathcal{A}$ is abelian.

The classification problem first appeared in the work of Arveson (1967) and it was subsequently investigated by Peters (1984), Hadwin and Hoover (1988), Power (1992) and Davidson and Katsoulis (2008), who finally settled affirmatively the case where $\mathcal{A}$ is abelian.

In the general case where $\mathcal{A}$ may not be abelian, initial consideration for the isomorphism problem was given for

In the general case where $\mathcal{A}$ may not be abelian, initial consideration for the isomorphism problem was given for

- automorphisms with full Connes spectrum, (Muhly and Solel, 2000)

In the general case where $\mathcal{A}$ may not be abelian, initial consideration for the isomorphism problem was given for

- automorphisms with full Connes spectrum, (Muhly and Solel, 2000)
- simple C^*-algebras, (Davidson and Katsoulis, 2008).

In the general case where $\mathcal{A}$ may not be abelian, initial consideration for the isomorphism problem was given for

- automorphisms with full Connes spectrum, (Muhly and Solel, 2000)
- simple C^*-algebras, (Davidson and Katsoulis, 2008).

Considerable progress was made by

- Davidson and Kakariadis, 2014

who resolved the problem for *isometric isomorphisms* and dynamical systems consisting of unital injective endomorphisms of unital C^*-algebras. Actually the work of Davidson and Kakariadis went well beyond systems consisting of injective endomorphisms.

Theorem (Davidson and Kakariadis, 2014)

Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be unital C^*-dynamical systems and assume that one of the following conditions holds

(1) *$\mathcal{A}$ has trivial center.*

(2) *$\mathcal{A}$ is abelian.*

(3) *$\mathcal{A}$ is finite, i.e., no proper isometries.*

(4) *$\alpha(\mathcal{A})'$ is finite.*

(5) *$\alpha(R_\alpha) = R_\alpha$, where $R_\alpha = \overline{\cup_{k\geq 1} \ker(\alpha^k)}$.*

(6) *$\alpha(R_\alpha^\perp) \subseteq R_\alpha^\perp$.*

If the semicrossed products $\mathcal{A} \times_\alpha \mathbb{Z}^+$ and $\mathcal{B} \times_\beta \mathbb{Z}^+$ are isometrically isomorphic then the corresponding C^*-dynamical systems are outer conjugate.

Conjecture (Davidson and Kakariadis)

The (isometric) isomorphism problem has a positive resolution for arbitrary unital C^**-dynamical systems.*

Conjecture (Davidson and Kakariadis)

The (isometric) isomorphism problem has a positive resolution for arbitrary unital C^**-dynamical systems.*

Problem: The above conjecture of Davidson and Kakariadis is a true conjecture!

Conjecture (Davidson and Kakariadis)

The (isometric) isomorphism problem has a positive resolution for arbitrary unital C^**-dynamical systems.*

Problem: The above conjecture of Davidson and Kakariadis is a true conjecture! In other words, there exists a dynamical system $(\mathcal{A}, \alpha)$ that does not satisfy any of the six conditions given earlier.

For the second conjecture, we claim that for each $i = 1, 2, \ldots, 6$, there exists a dynamical system $(\mathcal{A}_i, \alpha_i)$ that fails the corresponding condition (i) from the above list. It is easy to see then that the dynamical system $(\oplus_{i=1}^{6} \mathcal{A}_i, \oplus_{i=1}^{6} \alpha_i)$ will fail all six conditions.

For the second conjecture, we claim that for each $i = 1, 2, \ldots, 6$, there exists a dynamical system $(\mathcal{A}_i, \alpha_i)$ that fails the corresponding condition (i) from the above list. It is easy to see then that the dynamical system $(\oplus_{i=1}^{6}\mathcal{A}_i, \oplus_{i=1}^{6}\alpha_i)$ will fail all six conditions.

The existence of dynamical systems that do not satisfy anyone of the conditions

(1) $\mathcal{A}$ has trivial center.

(2) $\mathcal{A}$ is abelian.

(3) $\mathcal{A}$ is finite, i.e., no proper isometries.

s a trivial matter.

For condition

(4) $\alpha(\mathcal{A})'$ is finite.

let $\mathcal{H}$ be a separable Hilbert space and let $B(\mathcal{H}), K(\mathcal{H})$ denote the bounded and compact operators respectively acting on $\mathcal{H}$. Let

$$\mathcal{A}_4 = B(\mathcal{H}) \otimes K(\mathcal{H}) + \mathbb{C}(I \otimes I)$$

acting on $\mathcal{H} \otimes \mathcal{H}$ and let α_4 be the unital endomorphism of $\mathcal{A}$ defined as

$$\alpha_4(S + \lambda I \otimes I) = \lambda I \otimes I, \quad S \in B(\mathcal{H}) \otimes K(\mathcal{H}),\ \lambda \in \mathbb{C}.$$

The dynamical system $(\mathcal{A}_4, \alpha_4)$ fails properrty (4) from the above list.

For condition

(6) $\alpha(R_\alpha^\perp) \subseteq R_\alpha^\perp$, where $R_\alpha = \overline{\cup_{k\geq 1} \ker(\alpha^k)}$,

For condition

(6) $\alpha(R_\alpha^\perp) \subseteq R_\alpha^\perp$, where $R_\alpha = \overline{\cup_{k\geq 1} \ker(\alpha^k)}$,

let $\mathcal{H}$ and $K(\mathcal{H})$ be as in the previous paragraph and consider

$$\mathcal{A}_6 = (K(\mathcal{H}) + \mathbb{C}I) \oplus \mathbb{C}I$$

acting on $\mathcal{H} \oplus \mathcal{H}$

For condition
(6) $\alpha(R_\alpha^\perp) \subseteq R_\alpha^\perp$, where $R_\alpha = \overline{\cup_{k\geq 1} \ker(\alpha^k)}$,
let $\mathcal{H}$ and $K(\mathcal{H})$ be as in the previous paragraph and consider

$$\mathcal{A}_6 = (K(\mathcal{H}) + \mathbb{C}I) \oplus \mathbb{C}I$$

acting on $\mathcal{H} \oplus \mathcal{H}$ and let α_6 be the unital endomorphism of $\mathcal{A}_6$ defined as

$$\alpha_6(K + \lambda I, \mu I) = (\mu I, \lambda I), \quad K \in K(\mathcal{H}),\ \lambda, \mu \in \mathbb{C}.$$

For condition

(6) $\alpha(R_\alpha^\perp) \subseteq R_\alpha^\perp$, where $R_\alpha = \overline{\cup_{k\geq 1} \ker(\alpha^k)}$,

let $\mathcal{H}$ and $K(\mathcal{H})$ be as in the previous paragraph and consider

$$\mathcal{A}_6 = (K(\mathcal{H}) + \mathbb{C}I) \oplus \mathbb{C}I$$

acting on $\mathcal{H} \oplus \mathcal{H}$ and let α_6 be the unital endomorphism of $\mathcal{A}_6$ defined as

$$\alpha_6(K + \lambda I, \mu I) = (\mu I, \lambda I), \quad K \in K(\mathcal{H}),\ \lambda, \mu \in \mathbb{C}.$$

It is clear that $\ker(\alpha_6^k) = K(\mathcal{H}) \oplus 0$, for all $k \geq 1$, and so $R_\alpha = K(\mathcal{H}) \oplus 0$. Therefore $R_{\alpha_6}^\perp = 0 \oplus \mathbb{C}I$ and so

For condition
(6) $\alpha(R_\alpha^\perp) \subseteq R_\alpha^\perp$, where $R_\alpha = \overline{\cup_{k\geq 1} \ker(\alpha^k)}$,
let $\mathcal{H}$ and $K(\mathcal{H})$ be as in the previous paragraph and consider

$$\mathcal{A}_6 = (K(\mathcal{H}) + \mathbb{C}I) \oplus \mathbb{C}I$$

acting on $\mathcal{H} \oplus \mathcal{H}$ and let α_6 be the unital endomorphism of $\mathcal{A}_6$ defined as

$$\alpha_6(K + \lambda I, \mu I) = (\mu I, \lambda I), \quad K \in K(\mathcal{H}),\ \lambda, \mu \in \mathbb{C}.$$

It is clear that $\ker(\alpha_6^k) = K(\mathcal{H}) \oplus 0$, for all $k \geq 1$, and so $R_\alpha = K(\mathcal{H}) \oplus 0$. Therefore $R_{\alpha_6}^\perp = 0 \oplus \mathbb{C}I$ and so

$$\alpha_6(R_{\alpha_6}^\perp) = \mathbb{C}I \oplus 0 \not\subseteq 0 \oplus \mathbb{C}I = R_{\alpha_6}^\perp,$$

i.e., $(\mathcal{A}_6, \alpha_6)$ does not satisfy (6).

Note also that

$$\alpha_6(R_{\alpha_6}) = 0 \neq R_{\alpha_6}$$

and so $(\mathcal{A}_6, \alpha_6)$ does not satisfy

(5) $\alpha(R_\alpha) = R_\alpha$

as well.

Theorem (K. and Ramsey)

Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be two unital C^*-dynamical systems. Then $\mathcal{A} \rtimes_\alpha \mathbb{Z}^+$ and $\mathcal{B} \rtimes_\beta \mathbb{Z}^+$ are isometrically isomorphic if and only $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate.

Theorem (K. and Ramsey)

Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be two unital C^*-dynamical systems. Then $\mathcal{A} \rtimes_\alpha \mathbb{Z}^+$ and $\mathcal{B} \rtimes_\beta \mathbb{Z}^+$ are isometrically isomorphic if and only $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate.

Strategy of the proof: Davidson and Kakariadis noticed that if

$$\psi : \mathcal{A} \rtimes_\alpha \mathbb{Z}^+ \longrightarrow \mathcal{B} \rtimes_\beta \mathbb{Z}^+$$

is an isometric isomorphism so that the zero Fourier coefficient of $\psi(v)$ is very small (less than 0.15), then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate.

Theorem (K. and Ramsey)

Let $(\mathcal{A}, \alpha)$ *and* $(\mathcal{B}, \beta)$ *be two unital* C^**-dynamical systems. Then* $\mathcal{A} \rtimes_\alpha \mathbb{Z}^+$ *and* $\mathcal{B} \rtimes_\beta \mathbb{Z}^+$ *are isometrically isomorphic if and only* $(\mathcal{A}, \alpha)$ *and* $(\mathcal{B}, \beta)$ *are outer conjugate.*

Strategy of the proof: Davidson and Kakariadis noticed that if

$$\psi : \mathcal{A} \rtimes_\alpha \mathbb{Z}^+ \longrightarrow \mathcal{B} \rtimes_\beta \mathbb{Z}^+$$

is an isometric isomorphism so that the zero Fourier coefficient of $\psi(v)$ is very small (less than 0.15), then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate. We construct an automorhism ρ_ψ of $\mathcal{B} \rtimes_\beta \mathbb{Z}^+$ that $\rho_\psi \circ \psi$ satisfies that property.

A *multivariable* C^**-dynamical system* is a pair $(\mathcal{A}, \alpha)$ consisting of a unital C^*-algebra $\mathcal{A}$ along with unital $*$-endomorphisms $\alpha = (\alpha_1, \dots, \alpha_n)$ of $\mathcal{A}$ into itself.

A *multivariable* C^**-dynamical system* is a pair $(\mathcal{A}, \alpha)$ consisting of a unital C^*-algebra $\mathcal{A}$ along with unital $*$-endomorphisms $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathcal{A}$ into itself.

A row isometric covariant representation of $(\mathcal{A}, \alpha)$ consists of a non-degenerate $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a row isometry $V = (V_1, V_2, \ldots, V_n)$ acting on $\mathcal{H}^{(n)}$ so that $\pi(a)V_i = V_i\pi(\alpha(a))$, for all $a \in \mathcal{A}$ and $i = 1, 2, \ldots, n$.

A *multivariable* C^**-dynamical system* is a pair $(\mathcal{A}, \alpha)$ consisting of a unital C^*-algebra $\mathcal{A}$ along with unital $*$-endomorphisms $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathcal{A}$ into itself.

A row isometric covariant representation of $(\mathcal{A}, \alpha)$ consists of a non-degenerate $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a row isometry $V = (V_1, V_2, \ldots, V_n)$ acting on $\mathcal{H}^{(n)}$ so that $\pi(a)V_i = V_i\pi(\alpha(a))$, for all $a \in \mathcal{A}$ and $i = 1, 2, \ldots, n$.

The tensor algebra $\mathcal{T}^+(\mathcal{A}, \alpha)$ is the universal algebra generated by a copy of $\mathcal{A}$ and a row isometry v satisfying the covariance relations.

In the case where $\mathcal{A}$ is abelian, tensor algebras of multivariable systems were first studied in detail by Davidson and Katsoulis in 2011.

In the case where $\mathcal{A}$ is abelian, tensor algebras of multivariable systems were first studied in detail by Davidson and Katsoulis in 2011.

These authors developed a satisfactory dilation theory and provided invariants of a topological nature for algebraic isomorphisms, which in certain cases turned out to be complete.

The multivariable isomorphism problem

In the case where $\mathcal{A}$ is abelian, tensor algebras of multivariable systems were first studied in detail by Davidson and Katsoulis in 2011.

These authors developed a satisfactory dilation theory and provided invariants of a topological nature for algebraic isomorphisms, which in certain cases turned out to be complete.

Specifically, Davidson and Katsoulis utilized the concept of piecewise conjugacy for classical multivariable dynamical systems and they established that if the tensor algebras are algebraically isomorphic then the two dynamical systems must be piecewise conjugate.

In the case where $\mathcal{A}$ is abelian, tensor algebras of multivariable systems were first studied in detail by Davidson and Katsoulis in 2011.

These authors developed a satisfactory dilation theory and provided invariants of a topological nature for algebraic isomorphisms, which in certain cases turned out to be complete.

Specifically, Davidson and Katsoulis utilized the concept of piecewise conjugacy for classical multivariable dynamical systems and they established that if the tensor algebras are algebraically isomorphic then the two dynamical systems must be piecewise conjugate.
However, the converse could only be established for tensor algebras with $n = 2$ or 3 and some other cases.

Conjecture (Davidson and Katsoulis, 2011)

Let $(\mathcal{X}, \alpha)$ and $(\mathcal{Y}, \beta)$ be classical dynamical systems on compact Hausdorff spaces. Then $\mathcal{T}^+(C(\mathcal{X}), \alpha)$ and $\mathcal{T}^+(C(\mathcal{Y}), \beta)$ are isomorphic as algebras if and only if $(\mathcal{X}, \alpha)$ and $(\mathcal{Y}, \beta)$ are piecewise conjugate.

In order to study this conjecture Katsoulis Kakariadis utilized the framework of C^*-correspondences.

Muhly and Solel placed semicrossed products and (non-selfadjoint) graph algebras and their study into a much broader context.

Multivariable systems and tensor algebras

Muhly and Solel placed semicrossed products and (non-selfadjoint) graph algebras and their study into a much broader context.

Definition

If $(X, \mathcal{A})$ is a C^*-correspondence over a C^*-algebra $\mathcal{A}$, then the tensor algebra $\mathcal{T}^+(X, \mathcal{A})$ is the non-selfsadjoint norm-closed subalgebra of the Cuntz-Pimsner-Toeplitz algebra $\mathcal{T}(X, \mathcal{A})$ generated by the faithful copies of X and $\mathcal{A}$.

C^*-correspondence $(X, \mathcal{A})$:

C^*-correspondence $(X, \mathcal{A})$: X is a closed bimodule of $\mathcal{C} \subseteq B(\mathcal{H})$ satisfying

$$X^* X \subseteq \mathcal{A}$$

C^*-correspondence $(X, \mathcal{A})$: X is a closed bimodule of $\mathcal{C} \subseteq B(\mathcal{H})$ satisfying

$$X^* X \subseteq \mathcal{A}$$

Cuntz-Pimsner-Toeplitz algebra $\mathcal{T}(X, \mathcal{A})$:

C^*-correspondence $(X, \mathcal{A})$: X is a closed bimodule of $\mathcal{C} \subseteq B(\mathcal{H})$ satisfying

$$X^* X \subseteq \mathcal{A}$$

Cuntz-Pimsner-Toeplitz algebra $\mathcal{T}(X, \mathcal{A})$: the C^*-subalgebra of $B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$ generated by

$$\mathcal{A} \otimes I \text{ and } X \otimes S,$$

where S is the forward shift on $\ell^2(\mathbb{N})$.

C^*-correspondence $(X, \mathcal{A})$: X is a closed bimodule of $\mathcal{C} \subseteq B(\mathcal{H})$ satisfying

$$X^* X \subseteq \mathcal{A}$$

Cuntz-Pimsner-Toeplitz algebra $\mathcal{T}(X, \mathcal{A})$: the C^*-subalgebra of $B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$ generated by

$$\mathcal{A} \otimes I \text{ and } X \otimes S,$$

where S is the forward shift on $\ell^2(\mathbb{N})$.

$\mathcal{T}^+(X, \mathcal{A}) \subseteq \mathcal{T}(X, \mathcal{A})$ is generated by X and $\mathcal{A}$.

- Let (π, V) be a covariant representation of $(\mathcal{A}, \alpha)$ on $\mathcal{H}$ with π faithful.

- Let (π, V) be a covariant representation of $(\mathcal{A}, \alpha)$ on $\mathcal{H}$ with π faithful. Then the tensor algebra for the C^*-correspondence

$$\mathcal{A}_\alpha := (V\pi(\mathcal{A}), \pi(\mathcal{A}))$$

is completely isometrically isomorphic to $\mathcal{A} \rtimes_\alpha \mathbb{Z}^+$.

Some familiar examples

- Let (π, V) be a covariant representation of $(\mathcal{A}, \alpha)$ on $\mathcal{H}$ with π faithful. Then the tensor algebra for the C^*-correspondence

$$\mathcal{A}_\alpha := (V\pi(\mathcal{A}), \pi(\mathcal{A}))$$

is completely isometrically isomorphic to $\mathcal{A} \rtimes_\alpha \mathbb{Z}^+$.

- Let $(\pi, V_1, V_2, \dots, \mathcal{V}_n)$ be a covariant representation of the system $(\mathcal{A}, \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n))$ with π faithful. Then the tensor algebra for the C^*-correspondence

$$\mathcal{A}_\alpha := (\sum_{i=1}^{n} V_i\pi(\mathcal{A}), \pi(\mathcal{A}))$$

is completely isometrically isomorphic to $\mathcal{A} \rtimes_\alpha \mathbb{Z}^+$.

Conjecture (Muhly and Solel, 2000)

Consider C^**-correspondences* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$. *Then* $\mathcal{T}^+(X, \mathcal{A})$ *and* $\mathcal{T}^+(Y, \mathcal{B})$ *are completely isometrically isomorphic if and only if* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are unitarily equivalent, i.e., there exists a* $*$*-homomorphism* $\rho : \mathcal{A} \to \mathcal{B}$ *and* ρ*-unitary* $U : X \to Y$.

Conjecture (Muhly and Solel, 2000)

Consider C^**-correspondences* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$*. Then* $\mathcal{T}^+(X, \mathcal{A})$ *and* $\mathcal{T}^+(Y, \mathcal{B})$ *are completely isometrically isomorphic if and only if* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are unitarily equivalent, i.e., there exists a* $*$*-homomorphism* $\rho : \mathcal{A} \to \mathcal{B}$ *and* ρ*-unitary* $U : X \to Y$*.*

Lemma

If $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are unitarily equivalent, then* $\mathcal{T}^+(X, \mathcal{A})$ *and* $\mathcal{T}^+(Y, \mathcal{B})$ *are completely isometrically isomorphic.*

Lemma

If $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$ are the correspondences coming from multivariable systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ then $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$ are unitarily equivalent after a conjugation if there exists unitary matrix $U = [u_{i,j}] \in M_{n,m}(\mathcal{A})$ and $*$-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ so that

$$\begin{pmatrix} \alpha_1(a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n(a) \end{pmatrix} = U \begin{pmatrix} \gamma^{-1}\beta_1\gamma_1(a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma^{-1}\beta_m\gamma_1(a) \end{pmatrix} U^*,$$

for all $a \in \mathcal{A}$.

Theorem (Kakariadis and Katsoulis, 2014)

Under various conditions for automorphic unital C^**-dynamical systems, if* $\psi : \mathcal{T}^+(\mathcal{A}, \alpha) \to \mathcal{T}^+(\mathcal{B}, \beta)$ *is a completely isometric isomorphism then* $(\mathcal{A}, \alpha)$ *and* $(\mathcal{B}, \beta)$ *are unitarily equivalent after a conjugation.*

Theorem (Kakariadis and Katsoulis, 2014)

Under various conditions for automorphic unital C^*-dynamical systems, if $\psi : \mathcal{T}^+(\mathcal{A}, \alpha) \to \mathcal{T}^+(\mathcal{B}, \beta)$ is a completely isometric isomorphism then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are unitarily equivalent after a conjugation.

Theorem (Katsoulis and Ramsey, 2021)

If $\psi : \mathcal{T}^+(\mathcal{A}, \alpha) \to \mathcal{T}^+(\mathcal{B}, \beta)$ is a completely isometric isomorphism of unital C^*-dynamical systems, then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are unitarily equivalent after a conjugation.

Stable Isomorphisms of Operator Algebras

The motivation comes from

The motivation comes from

Theorem (Dor-On, Eilers and Geffen)

Let G_1, G_2 be row finite graphs and let $\mathcal{K}$ denote the compact operators. Then the following are equivalent:

(i) *The algebras $\mathcal{T}^+(G_1) \otimes K$ and $\mathcal{T}^+(G_2) \otimes K$ are completely isometrically isomorphic*

(ii) *The algebras $\mathcal{T}^+(G_1)$ and $\mathcal{T}^+(G_2)$ are completely isometrically isomorphic*

(iii) *The graphs G_1 and G_2 are isomorphic*

The motivation comes from

Theorem (Dor-On, Eilers and Geffen)

Let G_1, G_2 be row finite graphs and let $\mathcal{K}$ denote the compact operators. Then the following are equivalent:

(i) *The algebras $\mathcal{T}^+(G_1) \otimes K$ and $\mathcal{T}^+(G_2) \otimes K$ are completely isometrically isomorphic*

(ii) *The algebras $\mathcal{T}^+(G_1)$ and $\mathcal{T}^+(G_2)$ are completely isometrically isomorphic*

(iii) *The graphs G_1 and G_2 are isomorphic*

The equivalence of (ii) and (iii) is an earlier result of Katsoulis and Kribs (2004) and independently Solel (2004).

The motivation comes from

Theorem (Dor-On, Eilers and Geffen)

Let G_1, G_2 be row finite graphs and let $\mathcal{K}$ denote the compact operators. Then the following are equivalent:

(i) *The algebras $\mathcal{T}^+(G_1) \otimes K$ and $\mathcal{T}^+(G_2) \otimes K$ are completely isometrically isomorphic*

(ii) *The algebras $\mathcal{T}^+(G_1)$ and $\mathcal{T}^+(G_2)$ are completely isometrically isomorphic*

(iii) *The graphs G_1 and G_2 are isomorphic*

The equivalence of (ii) and (iii) is an earlier result of Katsoulis and Kribs (2004) and independently Solel (2004).

But the above Theorem is part of the program for resolving the Muhly Solel isomorphism problem and one needs to allow K to be any C^*-algebra!

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras with c_0-isomorphic diagonals and let $\mathcal{K}$ denote the compact operators. If $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isometrically isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras with c_0-isomorphic diagonals and let $\mathcal{K}$ denote the compact operators. If $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isometrically isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic.

Corollary

Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras with c_0-isomorphic diagonals and let $\mathcal{K}$ denote the compact operators. If $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are completely isometrically isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are completely isometrically isomorphic.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be analytic operator algebras with c_0-isomorphic diagonals and let $\mathfrak{K}_{\mathcal{A}}$ and $\mathfrak{K}_{\mathcal{B}}$ be operator algebras containing the compact operators. If $\mathcal{A} \otimes \mathfrak{K}_{\mathcal{A}}$ and $\mathcal{B} \otimes \mathfrak{K}_{\mathcal{B}}$ are isometrically isomorphic, then $\mathfrak{K}_{\mathcal{A}}$ and $\mathfrak{K}_{\mathcal{B}}$ are isometrically isomorphic and the algebras $\mathcal{A}$ and $\mathcal{B}$ are bicontinuously isomorphic.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be analytic operator algebras with c_0-isomorphic diagonals and let $\mathfrak{K}_{\mathcal{A}}$ and $\mathfrak{K}_{\mathcal{B}}$ be operator algebras containing the compact operators. If $\mathcal{A} \otimes \mathfrak{K}_{\mathcal{A}}$ and $\mathcal{B} \otimes \mathfrak{K}_{\mathcal{B}}$ are isometrically isomorphic, then $\mathfrak{K}_{\mathcal{A}}$ and $\mathfrak{K}_{\mathcal{B}}$ are isometrically isomorphic and the algebras $\mathcal{A}$ and $\mathcal{B}$ are bicontinuously isomorphic.

Corollary

Let G_1, G_2 be countable graphs and let $\mathcal{K}$ denote an operator algebra containing the compact operators. Then the following are equivalent:

(i) *The algebras $\mathcal{T}^+(G_1) \otimes K$ and $\mathcal{T}^+(G_2) \otimes K$ are isometrically isomorphic.*

(ii) *The algebras $\mathcal{T}^+(G_1)$ and $\mathcal{T}^+(G_2)$ are isomorphic as Banach algebras.*

(iii) *The graphs G_1 and G_2 are isomorphic.*

References

- P. Muhly and B. Solel, *Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes*, J. Funct. Anal. **158** (1998), 389–457.
- K. Davidson and E. Kakariadis, *Conjugate dynamical systems on C^*-algebras*, Inter. Math. Res. Not. (2014), 1289–1311.
- K. Davidson and E. Katsoulis, *Isomorphisms between topological conjugacy algebras*, J. Reine Angew. Math. **621** (2008), 29–51.
- K. Davidson and E. Katsoulis, *Semicrossed products of simple C*-algebras*, Math. Ann. **342** (2008), 515–525.
- K. Davidson and E. Katsoulis, *Operator algebras for multivariable dynamics*, Mem. Amer. Math. Soc. **209** (2011), no. 982, viii+53 pp.
- K. Davidson, O. Shalit and C. Ramsey, *The isomorphism problem for some universal operator algebras*, Advances in Mathematics **228**, (2011) 167–218

- A. Dor-On, S. Eilers and S. Geffen, *Classification of irreversible and reversible Pimsner operator algebras*, Compositio Mathematica, **156**, (2020) 2510–2535.
- D. Hadwin and T. Hoover, *Operator algebras and the conjugacy of transformations*, J. Funct. Anal. **77** (1988), 112–122.
- E. Kakariadis and E. Katsoulis, *Isomorphism invariants for multivariable* C^**-dynamics*,J. Noncommut. Geom. **8** (2014), 771–787.
- E. Katsoulis, *Non-selfadjoint operator algebras: dynamics, classification and C*-envelopes*, Recent Advances in Operator Theory and Operator Algebras, 27–81, CRC Press, Boca Raton, FL, 2018.
- E. Katsoulis and D. Kribs, *Isomorphisms of algebras associated with directed graphs*, Math. Ann. **330**, (2004), 709-728
- E. Katsoulis and C. Ramsey, *The isomorphism problem for tensor algebras of multivariable dynamical systems*, preprint arXiv:2101.07834.
- P. Muhly and B. Solel, *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc. (3) **81** (2000), 113–168.

References III

- P. Muhly and B. Solel, *Schur class operator functions and automorphisms of Hardy algebras*, Doc. Math. **13** (2008), 365–411.
- J. Peters, *Semicrossed products of C^*-algebras*, J. Funct. Anal. **59** (1984), 498–534.
- S.C. Power, *Classification of analytic crossed product algebras*, Bull. London Math. Soc. **24** (1992), 368–372.
- B. Solel, *You can see the arrows in a quiver operator algebras*, J. Aust. Math. Soc. **77** (2004), 111-122

Thank you Paul!!