Aperiodicity for Hilbert bimodules and C^*-inclusions

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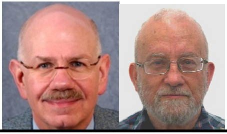
Based on

BKK, Ralf Meyer

(2018) "Aperiodicity, topological freeness and pure outerness"
(2020) "Noncommutative Cartan C*-subalgebras"
(2021) "Essential crossed products for inverse semigroup actions"
(2022) "Aperiodicity: the almost extension property and pseudo-expectations"



(2000) "On the Morita equivalence of tensor algebras"

Paul Muhly, Baruch Solel

1) Aperiodicity for Hilbert bimodules

Let $\alpha : A \to A$ be an automorphism of a C^*-algebra A.

Def.

- α is **purely outer** if $\alpha|_I$ is outer for every non-zero α-invariant $I \lhd A$
- α is **topologically free** if $\{[\pi] \in \widehat{A} : [\pi \circ \alpha] = [\pi]\}$ has empty interior in $\widehat{A}$ (O'Donovan 1975, Zeller-Meier 1968)

Thm. (Kishimoto 1982)

$$\underbrace{\left(\begin{array}{c} \Gamma_{\mathrm{Bor}}(\alpha|_I) \neq \{1\} \text{ for all} \\ \text{non-zero } \alpha\text{-invariant } I \lhd A \end{array} \right)}_{\text{spectrally non-trivial "Pasnicu-Phillips"}} \iff \underbrace{\left(\begin{array}{c} \forall_{b \in A} \forall_{0 \neq D \subseteq A \text{ hereditary}} \forall_{\varepsilon > 0} \exists_{a \in D_1^+} \\ \|\alpha(a) \cdot b \cdot a\| < \varepsilon \end{array} \right)}_{\text{aperiodic "Muhly-Solel"}}$$

Thm. (Pedersen-Olesen 1982) Assume A is separable

Then α is aperiodic $\iff$ α is topologically free and TFAE:

1. α^n is aperiodic for all $n > 0$
2. α^n is topologically free for all $n > 0$
3. A **detects ideals** in $A \rtimes_\alpha \mathbb{Z}$, i.e. $0 \neq I \lhd A \rtimes_\alpha \mathbb{Z} \Longrightarrow A \cap I \neq 0$.

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Ex. $\alpha : A \xrightarrow{\sim} A$ an automorphism $\Longrightarrow M_\alpha := A$ is a Hilbert A-bimodule with

$$x \cdot a := xa, \qquad a \cdot x := \alpha(a)x, \qquad \langle x, y\rangle_A := x^*y \qquad {}_A\langle x, y\rangle := \alpha^{-1}(xy^*)$$

Def. (Muhly-Solel 2000, K-Meyer) Let M be a normed A-bimodule

M is **aperiodic** if $\forall_{x\in M} \forall_{0\neq D\subseteq A \text{ hereditary}} \forall_{\varepsilon>0} \exists_{a\in D_1^+} \; \|a \cdot x \cdot a\| < \varepsilon$

Def.

M is a **Hilbert A-bimodule** if M is a right Hilbert A-module and a left Hilbert A-module such that ${}_A\langle x, y\rangle z = x\langle y, z\rangle_A$, $x, y, z \in M$.

Rem. If $\overline{\langle M, M\rangle_A} = {}_A\overline{\langle M, M\rangle} = A$, M is a Morita-Rieffel equivalence bimodule. In general, M induces a **partial homeomorphism** $\widehat{M}$ of $\widehat{A}$:

$$\widehat{A} \supseteq \widehat{\langle M, M\rangle_A} \xrightarrow{\widehat{M}} \widehat{{}_A\langle M, M\rangle} \subseteq \widehat{A}$$

where $\widehat{M}([\pi])$ acts on $M \otimes_\pi H_\pi$ by left multiplication on M.

Def. $I \triangleleft A$ is M-**invariant** if $MI = IM$. Then $M|_I := MI$ is a Hilbert I-bimodule

Ex. If $M_\alpha := A$ is associated to $\alpha : A \xrightarrow{\sim} A$, then $\widehat{M_\alpha}([\pi]) = [\pi \circ \alpha]$,

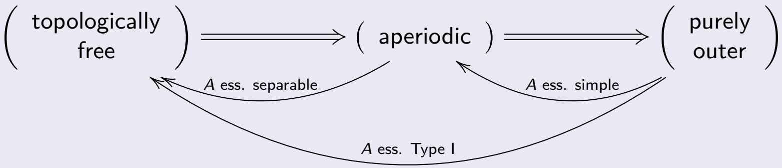
$M_\alpha I = IM_\alpha \iff \alpha(I) = I$, and then $M_\alpha|_I = M_{\alpha|_I}$

$\alpha(\cdot) = u(\cdot)u^*$ for a unitary $u \in \mathcal{M}(A) \iff M_\alpha \cong A$

Def. Let M be a **Hilbert A-bimodule**

- M is **purely outer** if $M|_I \not\cong I$ for every non-zero M-invariant $I \triangleleft A$
- M is **topologically free**: $\left\{[\pi] : \widehat{M}([\pi]) = [\pi]\right\}$ has empty interior in $\widehat{A}$
- M is **aperiodic**: $\forall_{x\in M} \forall_{0\neq D\subseteq A \text{ hereditary}} \forall_{\varepsilon>0} \exists_{a\in D_1^+} \ \|a\cdot x\cdot a\| < \varepsilon$

Thm. (K-Meyer 2018, 2022) For any Hilbert A-module we have



Thm. (K-Meyer 2018) Assume A **is ess. separable or ess. Type I**. TFAE:

1. $M^{\otimes n}$ is aperiodic for all $n > 0$
2. $M^{\otimes n}$ is topologically free for all $n > 0$
3. A **detects ideals** in $A \rtimes_M \mathbb{Z} = \overline{\bigoplus_{n\in\mathbb{Z}} M^{\otimes n}} = \mathcal{O}_M$

Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a **Fell bundle over a discrete group** G and let $A = B_1$.
$C_r^*(\mathcal{B}) = \overline{\bigoplus_{g\in G} B_g}$ is G-graded and there is a faithful $E : C_r^*(\mathcal{B}) \to A \subseteq C_r^*(\mathcal{B})$

Def. We say that $\mathcal{B} = \{B_g\}_{g\in G}$ is

- **purely outer**: if B_g is purely outer for every $g \in G \setminus \{1\}$
- **topologically free**: if B_g is topologically free for every $g \in G \setminus \{1\}$
- **aperiodic**: if B_g is aperiodic for every $g \in G \setminus \{1\}$

Def. Let $A \subseteq B$ be a C^*-inclusion.

- A **detects ideals in** B if $0 \neq I \triangleleft B \Longrightarrow A \cap I \neq 0$.
- A **supports** B if for every $b \in B^+ \setminus \{0\}$ there is $a \in A^+ \setminus \{0\}$ with $a \precsim b$

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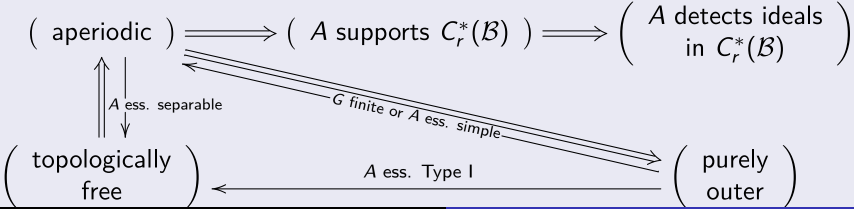
Thm.(K-Meyer 2018, 2020)



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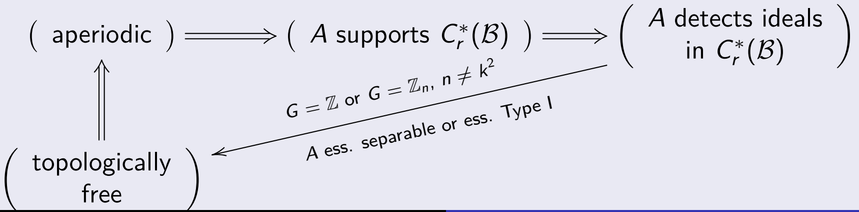

2) Aperiodic C^*-inclusions $A \subseteq B$

Def. (K-Meyer 2021)

$A \subseteq B$ is **aperiodic** if the Banach A-bimodule B/A is aperiodic.

Rem. If $E : B \to A$ a conditional expectation, then $B/A \cong \ker E$ and thus $A \subseteq B$ aperiodic $\iff \forall_{b\in \ker E}\, \forall_{0\neq D\subseteq A \text{ hereditary}}\, \forall_{\varepsilon>0}\, \exists_{a\in D_1^+}\ \|a\cdot b\cdot a\| < \varepsilon$

Ex. $A \subseteq C_r^*(\mathcal{B}) = \overline{\bigoplus_{g\in G} B_g}$ aperiodic $\iff \mathcal{B} = \{B_g\}_{g\in G}$ aperiodic

Def. (Pitts 2012) Let $I(A)$ be the injective envelope of A

Pseudo-expectation for $A \subseteq B$ is a ccp map $E : B \to I(A)$ with $E|_A = id$

Thm. (K-Meyer 2022)

If $A \subseteq B$ is aperiodic it admits a **unique pseudo-expectation** E and then

$E : B \to I(A)$ is faithful $\iff$ A supports all C with $A \subseteq C \subseteq B$

$\iff$ A detects ideals in all C with $A \subseteq C \subseteq B$

Thm. (K-Meyer 2022) Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a Fell bundle and $A := B_1$

Assume A is ess. separable or ess. Type I. TFAE:

(1) $A \subseteq C^*_r(\mathcal{B})$ aperiodic (equivalently $\mathcal{B}$ is aperiodic or topologically free)

(2) A supports all intermediate C^*-algebras $A \subseteq C \subseteq C^*_r(\mathcal{B})$

(3) A detects ideals in all intermediate C^*-algebras $A \subseteq C \subseteq C^*_r(\mathcal{B})$

(4) A detects ideals in $C := C^*_r(\{B_h\}_{h\in H})$ for all cyclic subgroups $H \subseteq G$

The above theorem extends to **Fell bundles over inverse semigroups** and in particular can be applied to **Fell bundles over étale groupoids**!!!

Ex. $(\mathcal{G},\Sigma)$ twisted étale groupoid with locally compact Hausdorff unit space X.

$C_0(X) \subseteq C_r^*(\mathcal{G},\Sigma)$ aperiodic $\iff \mathcal{G}$ is **topologically free**, i.e. for every open bisection $U \subseteq \mathcal{G} \setminus X$, the set $\{x \in X : \mathcal{G}(x) \cap U \neq \emptyset\}$ has empty interior in X.

Rem. There is always a pseudo-expectation induced by restriction $f \mapsto f|_X$

$$E_{ess} : C_r^*(\mathcal{G},\Sigma) \to \mathcal{B}(X)/\mathcal{M}(X) = I(C_0(X))$$

It descends to a faithful map on $C_{ess}^*(\mathcal{G},\Sigma) := C_r^*(\mathcal{G},\Sigma)/\mathcal{N}$ where $\mathcal{N} \subseteq \ker E_{ess}$

Def. (K-Meyer 2021)

We call $C_{ess}^*(\mathcal{G},\Sigma) = C_r^*(\mathcal{G},\Sigma)/\mathcal{N}$ the **essential groupoid algebra**.

Cor. TFAE:

1. $C_0(X) \subseteq C_{ess}^*(\mathcal{G},\Sigma)$ aperiodic
2. $\mathcal{G}$ is topologically free
3. $C_0(X)$ supports all intermediate C^*-algebras $C_0(X) \subseteq C \subseteq C_{ess}^*(\mathcal{G},\Sigma)$
4. $C_0(X)$ detects ideals in all intermediate C^*-algebras $C_0(X) \subseteq C \subseteq C_{ess}^*(\mathcal{G},\Sigma)$

3) Cartan C^*-inclusions $A \subseteq B$

Def. (Kumjian 1986) A C^*-inclusion $A \subseteq B$ is **regular** if it is nondegenerate and the normalizers $\{b \in B : bAb^* \subseteq A,\ b^*Ab \subseteq A\}$ generate B.

Prop. (Exel 2011) $A \subseteq B$ is regular $\iff$ $A = B_1$ and $B = \overline{\sum_{t \in S} B_t}$ is graded by a unital inverse semigroup S, i.e. $B_t^* = B_{t^*}$, $B_t \cdot B_s \subseteq B_{ts}$, and $B_t \subseteq B_s$ if $t \leq s$

Thm. (K-Meyer 2020+2022)

Assume $A \subseteq B$ is regular with a faithful cond. expectation $E : B \to A$ and $A = C_0(X)$ is **commutative**. TFAE:

1. A is a MASA in B, i.e. a **Cartan subalgebra** in the sense of Renault
2. $A \subseteq B$ is aperiodic
3. $A \subseteq B$ has a unique expectation (in fact a unique pseudo-expectation)
4. A supports all intermediate C^*-algebras $A \subseteq C \subseteq B$
5. A detects ideals in all intermediate C^*-algebras $A \subseteq C \subseteq B$
6. $B \cong C_r^*(\mathcal{G}, \Sigma)$ where $\mathcal{G}$ is étale, Hausdorff, top. free with the unit space X

Def. (Exel 2011) A **virtual commutant** of $A \subseteq B$ is an A-bimodule map $I \to B$ defined on an ideal $I \triangleleft A$. It is **trivial** if it has range in A.

Thm. (K-Meyer 2020)

Assume $A \subseteq B$ is regular with a faithful cond. expectation $E : B \to A$ and that $X := \mathrm{Prim}(A)$ is Hausdorff. TFAE:

1. $A \subseteq B$ is a **noncommutative Cartan inclusion** in the sense of Exel, i.e. it has no non-trivial virtual commutants
2. $A' \cap \mathcal{M}(B) = Z\mathcal{M}(A)$
3. $A \subseteq B$ has a unique conditional expectation
4. $B \cong C_r^*(\mathcal{G}, \mathcal{B})$ for a **purely outer Fell bundle** $\mathcal{B} = \{B_\gamma\}_{\gamma \in \mathcal{G}}$ **over a Hausdorff étale groupoid** $\mathcal{G}$ with unit space X and $A = C_r^*(X, \mathcal{B})$ (both $\mathcal{G}$ and $\mathcal{B}$ are uniquely determined by $A \subseteq B$)

If A is ess. Type I or ess. simple, then the above are equivalent to

5. $A \subseteq B$ is aperiodic.

Rem. $A \subseteq B$ noncommutative Cartan does not imply A detect ideals in B

Thm. Let $A \subseteq B$ be a regular C^*-inclusion with A **simple**. TFAE:

1. $A \subseteq B$ is a **C*-irreducible** (Rørdam 2021), i.e. all intermediate C^*-algebras $A \subseteq C \subseteq B$ are simple
2. $A' \cap \mathcal{M}(B) = \mathbb{C} \cdot 1$, i.e. $A \subseteq B$ is **irreducible**, and B is simple
3. there is a **unique faithful conditional expectation** $E : B \to A$
4. $A \subseteq B$ is a **noncommutative Cartan subalgebra** in the sense of Exel
5. $B \cong \mathrm{C}^*_{\mathrm{r}}(\mathcal{B})$ for an **outer Fell bundle** $\mathcal{B} = (B_g)_{g \in G}$ over a discrete group G with the unit fiber $B_1 = A$
   ($\mathcal{B}$ and G are uniquely determined by $A \subseteq B$)
6. $A \subseteq B$ is an **aperiodic inclusion** and the necessarily uni
   pseudo-expectation is faithful
7. $A \subseteq B$ **supports** all intermediate C*-algebras

Conjecture: If the above holds, then

every intermediate C^*-subalgebra $A \subseteq C \subseteq B$ is of the form $C = C^*_{\mathrm{r}}((B_h)_{h \in H})$ for a subgroup $1 \subseteq H \subseteq G$