Universal Toeplitz algebras and their boundary quotients

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$P =$ submonoid of a group G $(e \in P \subset G,\ P\,P \subset P)$

Left regular representation $p \mapsto L_p$ of P by isometries on $\ell^2(P)$:

$$L_p \delta_q = \delta_{pq} \qquad p \in P,$$

defined first on $\{\delta_q : q \in P\}$, then extended by linearity and continuity.

The (reduced) Toeplitz C*-algebra is the C*-algebra generated by L

$$\mathcal{T}_\lambda(P) := C^*(L_p : p \in P).$$

Since $\mathcal{T}_\lambda(P) \subset \mathcal{B}(\ell^2(P))$ one can use spatial techniques, but estimating norms of operators is not easy.
Given a collection $\{V_p\}_{p \in P}$, the question of whether whether $L_p \mapsto V_p$ produces a representation of $\mathcal{T}_\lambda(P)$ is a hard question:

$$\|f(V_p, V_p^* \mid p \in F)\| \overset{?}{\leqslant} \|f(L_p, L_p^* \mid p \in F)\|$$

three classical theorems

- (Coburn '67) S = unilateral shift and V = an isometry. Then the map $S^n \mapsto V^n$ $(n \in \mathbb{N})$ extends to a homomorphism $\pi_V : C^*(S) \to C^*(V)$, ... isomorphism iff $VV^* \neq 1$.

- (Douglas '72) Let Γ be a subgroup of $\mathbb{R}$. Suppose L is the l.r.r. and V is any isometric representation of $\Gamma^+ := \Gamma \cap [0, \infty)$. Then the map $L_p \mapsto V_p$ extends to a homomorphism $\pi_V : \mathcal{T}_\lambda(\Gamma^+) \longrightarrow C^*(V)$, ... isomorphism iff $V_p V_p^* \neq 1$ (for some, and hence all, $p \neq 0$).

- (Cuntz '81) Suppose L is the l.r.r. and V is an isometric representation of $P = \mathbb{F}_n^+$, the free monoid on n generators $\{1, 2, \cdots n\}$, and assume $\sum_{j=1}^n V_j V_j^* \leqslant 1$. Then the map $L_p \mapsto V_p$ extends to a homomorphism $\pi_V : \mathcal{T}_\lambda(\mathbb{F}_n^+) \longrightarrow C^*(V)$, ... isomorphism iff $\prod_{j=1}^n (1 - V_j V_j^*) \neq 0$.

presentations for $\mathcal{T}_\lambda(\mathbb{N})$, $\mathcal{T}_\lambda(\Gamma^+)$, and $\mathcal{T}_\lambda(\mathbb{F}_n^+)$

- Semigroup	Presentation	Properness
- $P = \mathbb{N}$;	$v^* v = 1$,	$1 - VV^* \neq 0$
- $P = \Gamma^+$;	$v_\gamma^* v_\gamma = 1, \quad v_\gamma v_\delta = v_{\gamma+\delta}$	$1 - V_p V_p^* \neq 0$
- $P = \mathbb{F}_n^+$;	$v_i^* v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$	$\prod_{j=1}^n (1 - V_j V_j^*) \neq 0$

Remarks:

1. universality of $\mathcal{T}_\lambda(P)$ (surprising for $\mathbb{F}_n^+$ because $\mathbb{F}_n$ is nonamenable)
2. uniqueness of $\mathcal{T}_\lambda(P)$ for 'jointly proper' representations
3. boundary quotient $\partial\mathcal{T}_\lambda(P)$ for 'maximally improper' representations

Constructible right ideals [Xin Li, 2012]

Definition/motivation by example:

Let's compute $L_p^* L_q L_r^* L_s$ for $p, q, r, s \in P$ with $p^{-1}qr^{-1}s = e$

$$(L_p^* L_q L_r^* L_s)\delta_x = L_p^* L_q L_r^* \delta_{sx} = \begin{cases} L_p^* L_q \delta_{r^{-1}sx} & \text{if } sx \in rP (\Leftrightarrow x \in s^{-1}rP) \\ 0 & \text{otherwise.} \end{cases}$$

Assuming $x \in s^{-1}rP$, we continue...

$$L_p^* L_q \delta_{r^{-1}sx} = \begin{cases} \delta_{p^{-1}qr^{-1}sx} & \text{if } x \in s^{-1}rq^{-1}pP \\ 0 & \text{otherwise.} \end{cases}$$

So $$L_p^* L_q L_r^* L_s \delta_x = \begin{cases} \delta_{(p^{-1}qr^{-1}s)x} & \text{if } x \in P \cap s^{-1}rP \cap s^{-1}rq^{-1}pP, \\ 0 & \text{otherwise.} \end{cases}$$

$K(p, q, r, s) := P \cap s^{-1}rP \cap s^{-1}rq^{-1}pP$ is a constructible right ideal.

Since $p^{-1}qr^{-1}s = e$, i.e., $\alpha = (p, q, r, s)$ is *neutral*, then

$$L_p^* L_q L_r^* L_s = \mathbb{1}_{K(p,q,r,s)}$$

Constructible right ideals: formal definition

Let $\mathcal{W}$ denote the set of all words in P of even length. For each $k \in \mathbb{N}$ and each word

$$\alpha = (p_1, p_2, \ldots, p_{2k}) \in \mathcal{W}(P)$$

we set

$$\dot{\alpha} := p_1^{-1} p_2 \cdots p_{2k-1}^{-1} p_{2k}, \qquad \tilde{\alpha} := (p_{2k}, p_{2k-1}, \ldots p_2, p_1) \quad \text{and}$$

$$K(\alpha) := P \cap (p_{2k}^{-1} p_{2k-1})P \cap (p_{2k}^{-1} p_{2k-1} p_{2k-2}^{-1} p_{2k-3})P \cap \cdots \cap (\dot{\tilde{\alpha}})P,$$

is a constructible right ideal.

Notice that $\dot{\tilde{\alpha}} := p_{2k}^{-1} p_{2k-1} \cdots p_2^{-1} p_1$.

$\mathcal{J}_P = \{K(\alpha) : \alpha \in \mathcal{W}, \dot{\alpha} = e\}$ is a semi-lattice under intersection.

universal Toeplitz C*-algebra $\mathcal{T}_u(P)$

Definition [L-Sehnem '21] Let $\mathcal{T}_u(P)$ be the universal C*-algebra with generators $\{t_p \colon p \in P\}$ such that (writing $\dot{t}_\alpha := t_{p_1}^* t_{p_2} \cdots t_{p_{2k-1}}^* t_{p_{2k}}$)

(T1) $t_e = 1$;

(T2) $\dot{t}_\alpha = 0$ if $K(\alpha) = \emptyset$ with $\dot{\alpha} = e$;

(T3) $\dot{t}_\alpha - \dot{t}_\beta = 0$ if $K(\alpha) = K(\beta)$ for α and β such that $\dot{\alpha} = e = \dot{\beta}$;

(T4) $\prod_{\beta \in F} (\dot{t}_\alpha - \dot{t}_\beta) = 0$ if $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$ for some α and finite set F with $\dot{\alpha} = e = \dot{\beta}$.

Definition: Universal Toeplitz and its diagonal:

$$\mathcal{T}_u(P) := C^*(\{t_p : p \in P\}) = \overline{\text{span}}\{\dot{t}_\alpha : \alpha \in \mathcal{W}\}$$

$$D_u := C^*(\{\dot{t}_\alpha \dot{t}_\alpha^* : \alpha \in \mathcal{W}\}) = \overline{\text{span}}\{\dot{t}_\alpha : \alpha \in \mathcal{W},\ \dot{\alpha} = e\}$$

Remark : Jack Spielberg has associated a couple of C*-algebras to each small category; one of them is isomorphic to $\mathcal{T}_u(P)$ under a somewhat different guise. Also cf. X. Li's booleanization of $\mathcal{J}_P$, C.F. Sehnem's product system construction.

some consequences

Relations (T1)–(T3) give Xin Li's semigroup C*-alg $C_s^*(P)$:

$\{t_p : \ p \in P\}$ is a semigroup of isometries generating $\mathcal{T}_u(P)$;

$\{\dot{t}_\alpha : \ \dot{\alpha} = e\}$ is a commuting family of projections generating D_u.

Moreover,

$$C_s^*(P) \xrightarrow{\pi} \mathcal{T}_u(P) \xrightarrow{\pi_L} \mathcal{T}_\lambda(P)$$

$$D_s \quad \xrightarrow{\pi|_D} \quad D_u \quad \xrightarrow{\cong} \quad D_\lambda$$

π_L is an isomorphism iff $E_u : \mathcal{T}_u(P) \to D_u$ is faithful (weak containment), e.g. if $P \hookrightarrow G$ amenable.

π and $\pi|_D$ are isomorphisms iff P satisfies independence.

independence: what it is and how it can fail

P satisfies independence iff any one of the following holds:

- $K(\alpha) = \bigcup_{\beta \in F} K(\beta) \implies K(\beta) = K(\alpha)$ for some $\beta \in F$.
- $\{\mathbb{1}_{K(\alpha)} : K(\alpha) \in \mathcal{J}\}$ is linearly independent.
- (T4) never applies beyond (T3).
- $C_s^*(P) \xrightarrow{\pi} \mathcal{T}_u(P)$ is an isomorphism
- $D_s \xrightarrow{\pi|_D} D_u$ is an isomorphism

Failures of independence:

Example 1 (Li '17) Independence fails on $\Sigma = \{0, 2, 3, \ldots\} \subset \mathbb{Z}$. because

$$K(3,2,2,3) = 2 + \mathbb{N} \text{ can be written as } (2+\Sigma) \cup (3+\Sigma).$$

So $D_s(\Sigma) \ncong D_\lambda(\Sigma)$ and $C_s^*(\Sigma) \ncong \mathcal{T}_\lambda(\Sigma)$.

Example 2 (L-Sehnem '21) Independence fails for all multiplicative monoids and all $ax + b$ monoids of nonmaximal orders $\mathcal{O}$ in number fields.

a partial action $G \circlearrowright D_\lambda$

[Li '17]: There is a partial action γ of G on D_λ such that if $p \in P$,

$$\gamma_p(\mathbb{1}_{K(\alpha)}) = \gamma_p(\dot{L}_\alpha) = \dot{L}_{(e,p,\alpha,p,e)} = L_p \dot{L}_\alpha L_p^* = \mathbb{1}_{pK(\alpha)},$$

$$\text{and} \qquad \mathcal{T}_\lambda(P) \;\cong\; D_\lambda \rtimes_{\gamma,r} G$$

[L-Sehnem '21]: $\qquad \mathcal{T}_u(P) \;\cong\; D_u \rtimes_\gamma G.$

This gives

$$D_u \rtimes_u G \cong \mathcal{T}_u(P) \xrightarrow{\pi_L} \mathcal{T}_\lambda(P) \cong D_\lambda \rtimes_r G$$

faithful representations of $\mathcal{T}_\lambda(P)$

Define $P^* := P \cap P^{-1}$ (the group of invertibles in P).

Theorem [Li '17]: When $P^* = \{e\}$, π is faithful iff $\pi|_{D_\lambda}$ is faithful.

When $P^* \neq \{e\}$ we should not expect this to be true (take $P = G$).

The partial action $G \circlearrowright D_\lambda$ restricts to an action $P^* \circlearrowright D_\lambda$ and

$$D_\lambda \rtimes_{\gamma,r} P^* \hookrightarrow D_\lambda \rtimes_{\gamma,r} G \cong \mathcal{T}_\lambda(P)$$

Theorem [L-Sehnem '21]: Every nontrivial ideal of $\mathcal{T}_\lambda(P)$ has nontrivial intersection with the subalgebra $D_\lambda \rtimes_{\gamma,r} P^*$.

Equivalently:

$$\pi : \mathcal{T}_\lambda(P) \to \mathcal{B}(\mathcal{H}) \text{ is faithful iff it is faithful on } D_\lambda \rtimes_{\gamma,r} P^*.$$

topological freeness and jointly proper isometries

When $P^* \circlearrowright D_\lambda$ is topologically free, the ideal intersection property drops down to D_λ. The key is a result of Archbold-Spielberg.

Definition: P satisfies (TopFree) if for every $u \in P^* \setminus \{e\}$ and every $\mathcal{C} \underset{\text{fin}}{\subset} \mathcal{J} \setminus \{P\}$, there exists $q \in P \setminus \bigcup_{R \in \mathcal{C}} R$ such that $uqP \neq qP$.

Definition: $\{W_p : p \in P\}$ is *jointly proper* if $\prod_{\alpha \in F} (I - \dot{W}_\alpha) \neq 0$ for every finite collection F of neutral words with $K(\alpha) \neq P$.

Theorem [L-Sehnem '21]: Suppose $E_u : \mathcal{T}_u(P) \to D_u$ is faithful, $P \hookrightarrow G$ satisfies (TopFree), and $\{W_p : p \in P\}$ satisfies (T1)–(T4). Then $L_p \mapsto W_p$ extends to a homomorphism

$$\mathcal{T}_\lambda(P) \xrightarrow{\pi_W} C^*(W),$$

which is an isomorphism if and only if W is jointly proper.

boundary quotient and covariance algebra

Let $\Omega_P := \operatorname{Spec} D_\lambda$.

Theorem [Li '17] (cf. L- Crisp '07) Ω_P has a smallest nonempty closed G-invariant subset $\partial\Omega_P$, and the (reduced) boundary is

$$\partial\mathcal{T}_\lambda(P) \;\cong\; C(\partial\Omega_P) \rtimes_r G$$

By analogy, there is a full boundary, given by

$$\partial\mathcal{T}_u(P) \cong C(\partial\Omega_P) \rtimes_u G$$

A presentation of the full boundary quotient can be obtained by adding more relations to the presentation of $\mathcal{T}_u(P)$.
The idea comes from Sehnem's covariance algebra for product systems, for the canonical product system with one-dimensional fibres associated to the monoid P.

foundation sets

Definition: A *foundation set for the constructible right ideal* $K(\alpha)$ is a finite collection $\{K(\beta) : \beta \in F\} \subset \mathcal{J}$ such that

$$K(\alpha) \supset \bigcup_{\beta\in F} K(\beta) \quad \text{and} \quad pP \cap \bigcup_F K(\beta) \neq \emptyset \text{ for all } p \in K(\alpha).$$

The foundation set $\{K(\beta) : \beta \in F\}$ is *proper* if $K(\alpha) \supsetneq \bigcup_{\beta\in F} K(\beta)$.

Sehnem's strong covariance ideal leads to boundary relations

(T5) $\prod_{\beta\in F}(\dot{w}_\alpha - \dot{w}_\beta) = 0$ for foundation sets (which we may assume are proper because 'improper ones' are covered by (T4)). Recall:

(T4) $\prod_{\beta\in F}(\dot{t}_\alpha - \dot{t}_\beta) = 0$ if $K(\alpha) = \bigcup_{\beta\in F} K(\beta)$ for α and $F \underset{\text{fin}}{\subset} P$

(T3) $\dot{t}_\alpha - \dot{t}_\beta = 0$ if $K(\alpha) = K(\beta)$ for α and β.

"there is no (T6)"

Lemma [L-Sehnem '21]: (T1)–(T5) is a maximal set of relations, i.e. the quotient of $\mathcal{T}_u(P)$ by any extra relation 'of the same kind' is trivial.

Proof: If $K(\alpha) \supset \bigcup_{\beta \in F} K(\beta)$ is not a foundation set, then $pP \cap \bigcup_{\beta \in F} K(\beta) = \varnothing$ for some $p \in K(\alpha)$, so $t_p t_p^* \leqslant \prod_{\beta \in F}(\dot{t}_\alpha - \dot{t}_\beta)$. If the product vanishes, then so does the isometry t_p. □

The full (universal) boundary quotient $\partial\mathcal{T}_u(P)$ is defined by:

Theorem [L-Sehnem '21] The following are canonically isomorphic:

1. the covar. alg. $\mathbb{C} \rtimes_{\mathbb{C}^P} P$ of the 1-dim'l product system over P;
2. the universal C*-algebra with presentation (T1)–(T5);
3. the full partial crossed product $C(\partial\Omega_P) \rtimes_u G$.

purely infinite simple reduced boundary quotients

Theorem [L-Sehnem '21]: TFAE

1. The monoid P satisfies condition (PI):
   $$\forall p \neq q \text{ in } P \;\; \exists s \in P \text{ such that } psP \cap qsP = \emptyset.$$
2. every proper ideal of $\partial\mathcal{T}_u(P)$ is contained in the kernel of the canonical map
   $$\partial\mathcal{T}_u(P) \to \partial\mathcal{T}_\lambda(P) = C(\partial\Omega_P) \rtimes_r G;$$
3. the partial action $G \circlearrowright \partial\Omega_P$ is topologically free;

Corollary [L-Sehnem '21]: Assume $P \neq \{e\}$.

If condition (PI) above holds, then $\partial\mathcal{T}_\lambda(P)$ is purely infinite simple. The converse holds whenever the boundary action satisfies weak containment (i.e. $\partial\mathcal{T}_u(P) \cong \partial\mathcal{T}_\lambda(P)$ via the canonical map).

pure infiniteness from $b + ax$ monoids of integral domains

Let R be an integral domain that is *not* a field and let $R \rtimes R^\times$ be the associated $b + ax$ monoid. So the multiplication is

$$(b, a)(d, c) = (b + ad, ac), \qquad b, d \in R,\ a, c \in R^\times.$$

From [Cuntz '08] and [Li '10] we know $\partial\mathcal{T}_\lambda(R \rtimes R^\times)$ is purely infinite simple (ring C*-algebras).

We recover this verifying directly that $P = R \times R^\times$ satisfies

(PI): $\forall p \neq q$ in P $\exists s \in P$ such that $psP \cap qsP = \emptyset$

Let $p = (b, a)$ and $q = (d, c)$ with $p \neq q$. We can reduce to $b \neq d$.

Case 1: $b - d \notin acR$. Set $s := (0, ac)$. Then $psP \cap qsP = \emptyset$ because, otherwise, $b - d \in acR$, contradicting the assumption.

Case 2: $b - d \in acR^\times$. Let $\bar{x} \in R^\times$ with $b - d = ac\bar{x}$. Let $r \in R^\times$ non-invertible and set $s := (0, ac\bar{x}r)$. Then $psP \cap qsP = \emptyset$ because, otherwise, r would be invertible, contradicting the assumption.

orders in number fields: uniqueness for $b + ax$ monoids

Definition: Let K be a number field of degree d and let $\mathcal{O}_K$ be the ring of integers of K (it is a $\mathbb{Z}$-module of rank d). An *order* in K is a subring $\mathcal{O} \subset \mathcal{O}_K$ that is free of full rank as a $\mathbb{Z}$-module.

[Li-Norling '16] showed that independence fails for $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$ which is a proper subring of the ring of integers of $\mathbb{Q}[\sqrt{-3}]$.

Proposition [L-Sehnem '21]: The monoids $\mathcal{O}^\times$ and $\mathcal{O} \rtimes \mathcal{O}^\times$ do not satisfy independence for every nonmaximal order $\mathcal{O}$ in a number field.

Theorem [L-Sehnem '21]: For every order in a number field, $\mathcal{T}_\lambda(\mathcal{O} \rtimes \mathcal{O}^\times)$ is universal and unique for jointly proper isometric representations satisfying (T1)-(T4).

Proof: The monoid $\mathcal{O} \rtimes \mathcal{O}^\times$ satisfies condition (TopFree).

Thank you!