Tensor Algebras as Algebras of Functions

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A Halmos Doctrine

"If you want to study a question about operators on infinite dimensional Hilbert spaces, first formulate it in the setting of finite dimensional spaces. Answer it there, and only then move on to the infinite dimensional setting."

Rings of Operators I, 1936

F. Murray and J. von Neumann

- The introduction is a quintessential example of Halmos's doctrine in action.

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.

- (18) = Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren.

Fast Forward to the early 90's

- Gabriel, P.; Roiter, A. V. **Representations of finite-dimensional algebras**. Springer-Verlag, Berlin, 1992. iv+177 pp. ISBN: 3-540-62990-4 16G10 (16-02)
- Key Notion: Spectroid–the categorification of matrix theory.
- M. Pimsner, **A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$**. Free probability theory (Waterloo, ON, 1995), 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997.

Hochschild

On the structure of algebras with nonzero radical, Bull. Am. Math. Soc. 53 (1947), 369-377

Theorem (Theorem 4.1)

Every f. d. algebra B over $\mathbb{C}$ is a quotient of a tensor algebra $T_A(E)$, where A is semi-simple and E is an $A-A$-bimodule.

Proof idea.

Let $R = \mathrm{Rad}(B)$. Set $A := B/R$ and $E := R/R^2$. Then B is a quotient of $T_A(E) := A \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$, $E^{\otimes n} := E \otimes_A E \otimes_A \cdots \otimes_A E$. The key points that make the obvious maps work are: 1) B contains a copy of A, unique up to inner isomorphism, and 2) In every bimodule over a semisimple algebra sub-bimodules are complemented. $\square$

Nesbitt and Scott

Some remarks on algebras over an algebraically closed field, Ann. Math. 44 (1943), 534–553

Theorem (Theorem 2.1)

Every f.d. algebra over $\mathbb{C}$ is Morita equivalent to an algebra that is commutative modulo its radical.

Remark

Morita equivalence was defined by K. Morita in **Duality for modules and its applications to the theory of rings with minimum condition**. *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 6 (1958), 83–142.*

Gabriel's Corollary

Unzerlegbare Darstellungen. I. Manuscripta Math. 6 (1972),71–103

Corollary

Every finite dimensional algebra over $\mathbb{C}$ is Morita equivalent to a quotient of the path algebra of the path category generated by a finite quiver (a.k.a. a finite directed graph).

Definition

A quiver: $\mathcal{Q} := \{\mathcal{Q}^0, \mathcal{Q}^1, r, s : \mathcal{Q}^1 \mapsto \mathcal{Q}^0\}$.

Remark (Second sentence.)

Für einen solchen 4-Tupel schlagen wir die Bezeichnung Köcher vor, und nicht etwa Graph, weil letzterem Wort schon zu viele verwandte Begriffe anhaften.

Google Translate

Remark (Second sentence)

"For such a 4-tuple we propose the notation quiver, and not graph, because the latter word is already used by too many people cling to terms."

Tensorial Polynomials and Series

- From now on A is a C^*-algebra.
- E will be a C^*-correspondence.
  - E is a bimodule over A.
  - E_A has an A-valued inner product $\langle \cdot, \cdot \rangle$.
  - The left action of A on E is given by a C^*-homomorphism $\varphi : A \to \mathcal{L}(E)$.
- Tensor powers: $E^{\otimes 0} := A$, $E^{\otimes 1} := E$, and $E^{\otimes n} := E \otimes_A E \otimes_A \cdots \otimes_A E$.
- The tensor algebra: $T_A(E) := A \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$, algebraic direct sum.
- The tensor series algebra: $T_A((E)) := A \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$, algebraic direct product.

Tensorial Polynomials and Series (cont.)

Convolution Product

- $(\sum_{k\geq 0} a_k)(\sum_{l\geq 0} b_l) = \sum_{n\geq 0} c_n$, where $c_n = \sum_{k+l=n} a_k \otimes b_l,\ a_k, b_k, c_k \in E^{\otimes k}$.

Tensorial Functions

The tensor algebra functor is the left adjoint of the forgetful functor.

- It is natural to try to realize a ring as a ring of functions on some collection of representations.
- The case for tensor algebras is "simplified" by this

Key Observation (A purely algebraic fact)

If $\rho : T_A(E) \to B(H)$ is an algebra homomorphism, then $\rho|A$ is a homomorphism of A to $B(H)$ and $\rho|E$ is a bimodule map: $\rho(a \cdot \xi \cdot b) = \rho(a)\rho(\xi)\rho(b)$. And conversely, each homomorphism of A into $B(H)$ together with a bimodule map of E into $B(H)$ can be extended uniquely to a homomorphism of $T_A(E)$ into $B(H)$.

A rich class of bimodule maps

Definition

A bimodule map of ${}_AE_A$ is a pair $(\pi, \mathfrak{Z})$, where π is a C^*-representation of A in $B(H_\pi)$ and $\mathfrak{Z} : E \to B(H_\pi)$ is a linear map that is continuous in norm (i.e. bounded) such that

$$\mathfrak{Z}(a \cdot \xi \cdot b) = \pi(a)\mathfrak{Z}(\xi)\pi(b), \qquad a, b \in A, \xi \in E.$$

Bimodule maps (cont.)

Definition (Induced Representations)

Given $\pi : A \to B(H_\pi)$, define $\pi^E \circ \varphi : A \to B(E \otimes_A H_\pi)$ by $\pi^E \circ \varphi(a) := \varphi(a) \otimes I_{H_\pi}$.

Lemma (JFA 158 (1998) Lemma 3.5.1)

If $(\pi, \mathfrak{Z})$ is a bimodule map with values in $B(H_\pi)$, then $\mathfrak{Z}$ is completely bounded, and if $\mathfrak{z} : E \otimes H_\pi \to H_\pi$ is defined by $\mathfrak{z}(\xi \otimes h) := \mathfrak{Z}(\xi)h$, then $\mathfrak{z}\pi^E \circ \varphi(\cdot) = \pi(\cdot)\mathfrak{z}$, i.e. $\mathfrak{z} \in \mathfrak{I}(\pi^E \circ \varphi, \pi) := E^\pi$, and $\|\mathfrak{Z}\|_{cb} = \|\mathfrak{z}\|$. Conversely, given $\mathfrak{z} \in E^\pi$, the formula $\mathfrak{Z}(\xi)h := \mathfrak{z}(\xi \otimes h)$ defines uniquely a bimodule map $(\pi, \mathfrak{Z})$.

Immediate Objective

GOAL

To show how to represent $T_A(E)$ as a space of polynomial functions on $\coprod_{\pi \in \widehat{A}^0} E^\pi$, where $\widehat{A}$ is the category whose objects, $\widehat{A}^0$, are the representations of A and whose morphisms are intertwiners, $\mathfrak{I}(\pi, \sigma)$.

Exercises

1. *Does $\pi \mapsto \pi^E \circ \varphi$ induce an endofunctor of $\widehat{A}$?*
2. *If so, what are the natural transformations between it and the identity functor on $\widehat{A}$? If not, what else can you say?*

The representation $\pi \times \mathfrak{z}$ of $T_A(E)$.

Definition

For $\pi \in \widehat{A}$ and $\mathfrak{z} \in E^{\pi}$, $\pi \times \mathfrak{z}$ is the representation of $T_A(E)$ on H_π defined on generators by:

$$\pi \times \mathfrak{z}(a)(h) := \pi(a)h, \qquad a \in A, h \in H_\pi,$$

and

$$\pi \times \mathfrak{z}(\xi)(h) := \mathfrak{z}(\xi \otimes h), \qquad \xi \in E, h \in H_\pi.$$

$\pi \times \mathfrak{z}(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k) = ?$

$$\begin{aligned} \pi \times \mathfrak{z}(\xi_1 \otimes \xi_2 \otimes \cdots \xi_k)(h) &= (\pi \times \mathfrak{z})(\xi_1) \cdots (\pi \times \mathfrak{z})(\xi_k)(h) \\ &= \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes(k-1)}} \otimes \mathfrak{z})(\xi_1 \otimes \xi_2 \otimes \cdots \xi_k \otimes h) \\ &= \mathcal{Z}_k(\mathfrak{z})(\xi_1 \otimes \xi_2 \otimes \cdots \xi_k \otimes h), \end{aligned}$$

where $\mathcal{Z}_k : E^\pi \to B(E^{\otimes k} \otimes H, H)$ is defined by $\mathcal{Z}_k(\mathfrak{z}) := \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes(k-1)}} \otimes \mathfrak{z})$, $k > 0$, and $\mathcal{Z}_0(\mathfrak{z}) \equiv I_{H_\pi}$. Thus $\pi \times \mathfrak{z}$ maps $E^{\otimes k}$ to $B(H_\pi)$ and is given by $\pi \times \mathfrak{z}(\theta)(h) = \mathcal{Z}_k(\mathfrak{z})(\theta \otimes h) := \mathcal{Z}_k(\mathfrak{z})L^{(k)}(\theta)(h)$, where $\theta \in E^{\otimes k}$, $h \in H_\pi$ and $L^{(k)}(\theta)(h) := \theta \otimes h$.

Polynomial Functions

Definition

For $F \in T_A(E)$, $\widehat{F} : \coprod_{\pi \in \widehat{A}^0} E^\pi \to \coprod_{\pi \in \widehat{A}^0} B(H_\pi)$ is defined to be $\widehat{F}(\pi \times \mathfrak{z}) := \pi \times \mathfrak{z}(F) = \sum_{k \geq 0} \mathcal{Z}_k(\mathfrak{z}) L^{(k)}(\theta_k)$, where $F = \sum_{k \geq 0} \theta_k$, $\theta_k \in E^{\otimes k}$.

Notation

$\widehat{F} = \{\widehat{F}_\pi\}_{\pi \in \widehat{A}^0}$, $\widehat{F}_\pi(\mathfrak{z}) = \widehat{F}(\pi \times \mathfrak{z})$.

Theorem

The map $F \to \widehat{F}$ is a faithful homomorphism of $T_A(E)$ into $\coprod_{\pi \in \widehat{A}^0} PolyMaps(E^\pi, B(H_\pi))$. The image is denoted $\widehat{T_A(E)}$.

The Derivative

- $\mathcal{Z}_{k+l}(\mathfrak{z}) = \mathcal{Z}_k(\mathfrak{z})(I_{E^{\otimes k}} \otimes \mathcal{Z}_l(\mathfrak{z}))$, $\mathfrak{z} \in E^\pi$.
- $(D\mathcal{Z}_k)(\mathfrak{z})[\zeta] = \sum_{l=0}^{k-1} \mathcal{Z}_l(\mathfrak{z})(I_{E^{\otimes l}} \otimes \zeta)(I_{E^{\otimes(l+1)}} \otimes \mathcal{Z}_{k-l-1}(\mathfrak{z}))$, $\mathfrak{z}, \zeta \in E^\pi$.
- $D\widehat{F}(\pi, \mathfrak{z})[\zeta] = \sum_{k \geq 0} D(\mathcal{Z}_k(\mathfrak{z}))[\zeta] L^{(k)}(\theta_k)$.

What sort of structure does $\widehat{T_A(E)}$ impose upon $\coprod_{\widehat{A}^0} E^\pi$?

Answer (Incomplete but Heuristic)

- *Sets of the form* $\mathcal{U}(F) := \{(\pi, \xi) \mid \|\widehat{F}(\pi, \xi)\|_{B(H_\pi)} < 1\}$ *should be open – a weak-$*$ like topology.*
- *"Closed under direct sums": If* $(\pi, \xi), \xi \in E^\pi$ *and* $(\sigma, \eta), \eta \in E^\sigma$ *are in an "open set"* $\mathcal{U}$, *then* $(\pi \oplus \sigma, \xi_\pi \oplus \eta_\sigma)$ *should be in* $\mathcal{U}$.
- *"Discs" are sufficient (?).*

Matricial Sets

Definition

$\mathcal{U} := \{\mathcal{U}(\pi)\}_{\pi\in\widehat{A^0}} \subseteq \coprod_{\pi\in\widehat{A^0}} E^\pi$ **respects direct sums** in case for all $\pi, \tau \in \widehat{A^0}$,

$$\mathcal{U}(\pi)\oplus\mathcal{U}(\tau) = \begin{bmatrix}\mathcal{U}(\pi) & 0\\ 0 & \mathcal{U}(\tau)\end{bmatrix}$$
$$\stackrel{\text{def}}{=} \left\{\begin{bmatrix}\mathfrak{z} & 0\\ 0 & \mathfrak{w}\end{bmatrix} \mid \mathfrak{z}\in\mathcal{U}(\pi), \mathfrak{w}\in\mathcal{U}(\tau)\right\} \subseteq \mathcal{U}(\pi\oplus\tau).$$

We then say $\mathcal{U}$ is a **matricial set** (which is called **open** iff each $\mathcal{U}(\pi)$ is).

Matricial Discs

- A (disc) **center** (or a **center field**) is a family $\zeta = \{\zeta_\pi\}_{\pi \in \widehat{A}^0}$ such that $\zeta_{\pi \oplus \sigma} = \zeta_\pi \oplus \zeta_\sigma$.
- If $\zeta = \{\zeta_\pi\}_{\pi \in \widehat{A}^0}$ is a center and if $0 \leq R \leq +\infty$ then the **matricial disc** centered at ζ of radius R, $\mathbb{D}(\zeta, R) := \coprod_{\pi \in \widehat{A}^0} \mathbb{D}(\zeta_\pi, R) \subseteq \coprod_{\pi \in \widehat{A}^0} E^\pi$.

Cauchy-Hadamard Theorem

Let $\Theta \sim \sum_{k\geq 0} \theta_k \in T_A((E))$, and let $R(\Theta) := \left(\limsup_{k\to\infty} \|\theta_k\|^{\frac{1}{k}}\right)^{-1}$. Then for each $\pi \in \widehat{A}^0$ and each $\zeta_\pi \in E^\pi$ the series

$$\widehat{\Theta}_\pi(\mathfrak{z}) := \sum_{k\geq 0} \mathcal{Z}_k(\mathfrak{z} - \zeta_\pi) L^{(k)}(\theta_k)$$

converges in operator norm on $B(H_\pi)$ for all $\mathfrak{z} \in \mathbb{D}(\zeta_\pi, R(\Theta))$ to a function $\widehat{\Theta}_\pi$ that is holomorphic from $\mathbb{D}(\zeta_\pi, R(\Theta))$ to $B(H_\pi)$. The series converges uniformly on proper subdiscs and $\widehat{\Theta}_\pi$ has the evident term-by-term derivative.

Local Uniform Boundedness

Definition

Let $f = \{f_\pi\}_{\pi \in \widehat{A}^0}$ be a family of functions defined on a matricial set $\mathcal{U}$ with $f_\pi : \mathcal{U}(\pi) \to B(H_\pi)$. Then f is called **locally uniformly bounded** in case for each center $\zeta \in \mathcal{U}$ there is a matricial disc $\mathbb{D}(\zeta, r) \subseteq \mathcal{U}$ such that

$$\sup_{\pi \in \widehat{A}^0} \sup_{\mathfrak{z} \in \mathbb{D}(\zeta_\pi, r)} \|f_\pi(\mathfrak{z})\|_{B(H_\pi)} < \infty.$$

Matricial Functions

Definition

Let $\mathcal{U} = \{\mathcal{U}(\pi)\}_{\pi \in \widehat{A}^0}$ be a matricial set, and let $\{f_\pi\}_{\pi \in \widehat{A}^0}$ be a family of functions, $f_\pi : \mathcal{U}(\pi) \to B(H_\pi)$.Then $\{f_\pi\}_{\pi \in \widehat{A}^0}$ is called a **matricial family of functions on $\mathcal{U}$** in case

$$\mathfrak{I}(\pi \times \mathfrak{z}, \tau \times \mathfrak{w}) \subseteq \mathfrak{I}(f_\pi(\mathfrak{z}), f_\tau(\mathfrak{w})),$$

$\mathfrak{z} \in \mathcal{U}(\pi), \mathfrak{w} \in \mathcal{U}(\tau)$. We also say $\{f_\pi\}_{\pi \in \widehat{A}^0}$ **respects (or preserves) intertwiners**.

Tensorial Taylor Series à la Joe Taylor

V. Vinnikov: "N.C. Cauchy-Riemann Equations"

Theorem (IEOT 76 (2013), Theorem 5.1)

Let $f = \{f_\pi\}_{\pi \in \widehat{A}^0}$ be a family of functions defined on a matricial set $\mathcal{U}$ such that $f_\pi : \mathcal{U}(\pi) \to B(H_\pi)$. Then f is a locally uniformly bounded matricial function if and only if for each matricial disc $\mathbb{D}(\zeta, r) \subseteq \mathcal{U}$ there is a tensor series $\Theta \in T_A((E))$, with $R(\Theta) \geq r$, such that

$$f_\pi(\mathfrak{z}) = \widehat{\Theta}_\pi(\mathfrak{z}), \qquad \mathfrak{z} \in \mathbb{D}(\zeta_\pi, r), \pi \in \widehat{A}^0.$$

The $\mathcal{Z}$-Transform

Definition

Let $\Theta \in T_A((E))$ be a tensor series. The $\mathcal{Z}$-transform of Θ, $\widehat{\Theta}(\cdot)$, is the formal sum $\sum_{k\geq 0} \mathcal{Z}_k(\cdot)L^{(k)}(\theta_k) := \mathcal{Z}\Theta(\cdot)$.

Proposition (de Moivre 1730)

$$\widehat{\Theta}\widehat{\Psi} = \mathcal{Z}(\Theta * \Psi)$$

Thank You