Regularity Properties of Piecewise Linear Maps

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Piecewise linear maps come up when using finite element methods to numerically solve PDEs, in approximation theory, and recently in study of ReLU neural networks which are piecewise linear. The purpose of this short project is to study some properties of these maps which to my knowledge have not yet been investigated.

Let's gives some quick definitions: A convex polyhedron is a set of the form

$$C = \{x \in \mathbb{R}^d | \langle c_i, x \rangle - b_i = 0, i = 1, \ldots, m\}.$$

Examples include cubes in arbitrary dimensions, squares, triangles etc. We say that f is piecewise linear subordinate to a finite collection of convex polyhedra $C_1, \ldots, C_m \subseteq \mathbb{R}^d$, if the restriction of f to each C_i is an affine function.

Here are some questions related to pieceswise linear functions, which we could touch upon in a short week project. The idea is that we would work on the second question about f being Lipschitz, but if we find it too easy or too hard we could move to something else.

Questions:

1. Show that if f is piecewise linear as defined above then it is continuous (this isn't too hard).
2. Is f necessarily Lipschitz?
3. If f is one-to-one on the union of the C_i, is it a homeomorphism onto its image? (Recall that in general a one-to-one continuous mapping may not be a homeomorphism).
4. Can f be extended to a piecewise linear function on all of $\mathbb{R}^d$?

The plan would be for us to meet all or most days for an hour or so to discuss current thoughts and futures steps. In some days this may be on Zoom. The rest of the time will be devoted to independent research (without my supervision).