Cartan subalgebras of twisted groupoid C^*-algebras including higher rank graph C^*-algebras

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The big picture

Outline

1. Introduction to Cartan subalgebras of C^*-algebras
2. Theorem 1: Cartan subalgebras generated by subgroupoids
   - Joint work with Duwenig, Gillaspy, Reznikoff, and Wright
   - https://arxiv.org/pdf/2001.08270.pdf
3. Theorem 2: Weyl construction for our Cartan pairs
   - Joint work with Duwenig and Gillaspy
   - https://arxiv.org/pdf/2010.04137.pdf
4. Application of Theorem 1 to higher rank graph C^*-algebras
   - Joint work with Reznikoff and Wright

C^*-algebra

Definition (Gelfand-Naimark Theorem)

A C^*-**algebra** A is a norm-closed $*$-subalgebra of the algebra of bounded linear operators $B(H)$ for some Hilbert space H.

Examples

- $M_n(\mathbb{C})$
- $C_0(X)$, where X is a locally compact Hausdorff space

Cartan subalgebra of a C^*-algebra

Definition

Let A be a C^*-algebra. We say $B \subseteq A$ is a **Cartan subalgebra** if

1. B is a maximal abelian subalgebra of A (MASA).
2. There exists a faithful conditional expectation $\Phi : A \to B$
   - Φ is contractive and linear
   - $\Phi|_B = \mathrm{id}_B$
   - $\Phi(a^* a) = 0 \implies a = 0$
3. The normalizer of B,

$$N(B) := \{n \in A \mid nbn^*, n^* bn \in B \quad \forall b \in B\}$$

   generates A as a C^*-algebra.

4. B contains an approximate identity for A.

Cartan subalgebra of a C^*-algebra

Example

Let $A := M_n(\mathbb{C})$. The collection B of diagonal matrices is Cartan.

1. maximal abelian subalgebra
2. $\Phi : M_n(\mathbb{C}) \to B$, $\Phi([x_{ij}]) = [\delta_{i=j} x_{ij}]$
3. $E_{ij} \in N(B)$ for all i, j, so $N(B)$ generates A.
4. B contains the identity matrix.

Remarks about Cartan subalgebras

- Cartan subalgebras of von Neumann algebras were originally defined by Vershik in 1971 (then by Feldman and Moore in 1977).
- Cartan subalgebras of C^*-algebras were defined by Renault in 1980.
- The Cartan subalgebra B is generally much simpler than A but can reveal information about A.
  - For a specific Cartan subalgebra B, if a representation of A is injective on B, then it is injective on A (Brown-Nagy-Reznikoff-Sims-Williams 2016).
  - For certain C^*-algebras, the presence of a Cartan subalgebra is equivalent to satisfying the Universal Coefficient Theorem, which implies that A can be classified by its Elliott invariant (Li 2019).

We are interested in the existence of Cartan subalgebras in a specific type of C^*-algebra, one that is built from a groupoid and a cocycle.

Ingredients for $C_r^*(\mathcal{G}, c)$

Intuitively, a **groupoid** $\mathcal{G}$ is a generalization of a group in which

- every element has an inverse
- multiplication is only defined on a subset $\mathcal{G}^{(2)}$ of $\mathcal{G} \times \mathcal{G}$

There are two "identity" elements associated with each $\gamma \in \mathcal{G}$, the **source** $\gamma^{-1}\gamma$ and the **range** $\gamma\gamma^{-1}$. The set of all identity elements is called the **unit space**, $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} \mid \gamma \in \mathcal{G}\}$. The **isotropy subgroupoid** $\text{Iso}(\mathcal{G}) = \{\gamma \in \mathcal{G} \mid \gamma^{-1}\gamma = \gamma\gamma^{-1}\}$.

In this talk, we will assume all groupoids are equipped with a locally compact Hausdorff topology, étale, and second countable.

Groupoid

Examples

- groups
- ordered pairs on a set
- equivalence relations: $(x, y) \in \mathcal{G} \iff x \sim y \in R$
- given any directed graph, we can construct the free groupoid generated by the edges

Ingredients for $C_r^*(\mathcal{G}, c)$ continued

A **cocycle** is a function $c : \mathcal{G}^{(2)} \to \mathbb{T}$ that satisfies the condition

$$c(\alpha, \beta\gamma)c(\beta, \gamma) = c(\alpha\beta, \gamma)c(\alpha, \beta).$$

In this talk, all cocycles are continuous.

Reduced Twisted Groupoid C^*-algebra $C_r^*(\mathcal{G}, c)$

Given a groupoid $\mathcal{G}$ and a cocycle c on $\mathcal{G}$, we denote by $C_c(\mathcal{G}, c)$ the collection of continuous compactly supported functions from $\mathcal{G}$ to $\mathbb{C}$ equipped with a convolution and an involution:

- $f * g(\gamma) := \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)c(\alpha, \beta)$
- $f^*(\gamma) := \overline{f(\gamma^{-1})c(\gamma, \gamma^{-1})}$.

The **reduced twisted groupoid C^*-algebra** $C_r^*(\mathcal{G}, c)$ is the completion of $C_c(\mathcal{G}, c)$ with respect to a certain norm.

Motivating Example

- group(oid) $(\mathbb{Z}^2, +)$
- cocycle $c_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$, $c_\theta((m,n),(p,q)) = e^{2\pi i n p \theta}$, where $\theta \in [0,1)$ is irrational
- $C_r^*(\mathbb{Z}^2, c_\theta)$ is called the **irrational rotation algebra**
- $C_r^*(\mathbb{Z} \times \{0\}, c_\theta)$ is a Cartan subalgebra of $C_r^*(\mathbb{Z}^2, c_\theta)$

Question: What are sufficient conditions on a subgroupoid $\mathcal{S}$ of $\mathcal{G}$ so that $C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $C_r^*(\mathcal{G}, c)$?

A first attempt

Question: What are sufficient conditions on a subgroupoid $\mathcal{S}$ of $\mathcal{G}$ so that $B := C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $A := C_r^*(\mathcal{G}, c)$?

Recall: $B \subseteq A$ is **Cartan** if

1. B is a MASA
   1.1 subalgebra
   1.2 abelian
   1.3 maximal abelian
2. $\exists$ faithful conditional expectation $\Phi : A \to B$
3. normalizer of B generates A
4. B contains an approximate identity for A

A necessary assumption

Definition

We say $\mathcal{S}$ is **immediately centralizing** if whenever $\gamma \in \mathsf{Iso}(\mathcal{G})$ commutes with uniformly bounded powers of every element of $\mathcal{S}$, then γ commutes with every element of $\mathcal{S}$.

Examples

1. If $\mathsf{Iso}(\mathcal{G})$ is abelian, then $\mathcal{S}$ is immediately centralizing.
2. If $\mathcal{S}$ has the **unique root property**, i.e.,

$$\alpha^k = \beta^k \implies \alpha = \beta$$

   then $\mathcal{S}$ is immediately centralizing.

Our final answer

Theorem 1 (Duwenig-Gillaspy-N.-Reznikoff-Wright 2020)

Let $\mathcal{G}$ be a second countable, locally compact Hausdorff, étale groupoid, and let c be a continuous cocycle on $\mathcal{G}$. Suppose $\mathcal{S}$ is maximal among abelian subgroupoids of $\mathrm{Iso}(\mathcal{G})$ on which c is symmetric. If $\mathcal{S}$ is open, closed, normal, and immediately centralizing, then $C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $C_r^*(\mathcal{G}, c)$.

Answer to a similar question

Theorem 1 (Duwenig-Gillaspy-N.-Reznikoff-Wright 2020)

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Theorem (Brown-Nagy-Reznikoff-Sims-Williams 2016)

Let $\mathcal{G}$ be a locally compact Hasudorff étale groupoid. If $\mathrm{Iso}(\mathcal{G})^\circ$ is abelian and closed, then $C_r^*(\mathrm{Iso}(\mathcal{G})^\circ)$ is Cartan in $C_r^*(\mathcal{G})$.

Theorem 1 in action

Consider $\mathcal{G} = \mathbb{Z}^5$ with multiplication

$$(a_1, a_2, a_3, a_4, a_5) * (b_1, b_2, b_3, b_4, b_5)$$
$$= (a_1 + b_1 + 2a_5 b_3, a_2 + b_2 + 2a_5 b_4, a_3 + b_3, a_4 + b_4, a_5 + b_5)$$

and cocycle

$$\mathbf{c}((a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5)) = (-1)^{a_4 b_1}$$

Note:

- $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}$ because $\mathcal{G}$ is a group.
- $\mathcal{G}$ has the unique root property, so any subgroup is immediately centralizing.

We must find maximal abelian subgroups on which $\mathbf{c}$ is symmetric, then check that they are open, closed, and normal.

Theorem 1 in action

Maximal abelian subgroups on which **c** is symmetric:

$\mathcal{S}_0 = \mathbb{Z} \times \mathbb{Z} \times \{0\} \times \{0\} \times \mathbb{Z}$
$\mathcal{S}_1 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times 2\mathbb{Z} \times \{0\}$
$\mathcal{S}_2 = 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \{0\}$

They are all open, closed, and normal, so Theorem 1 implies $C_r^*(\mathcal{S}_i)$ is Cartan in $C_r^*(\mathcal{G}, \mathbf{c})$.

There's more

In Theorem 1, we have $C_r^*(\mathcal{S}, c)$ is Cartan in $C_r^*(\mathcal{G}, c)$.

Theorem (Renault 2008)

Suppose A is a C^*-algebra and $B \subseteq A$ is Cartan. There exists a topologically principal étale groupoid $\mathcal{W}$ and a twist Σ over $\mathcal{W}$ such that

$$A \cong C_r^*(\mathcal{W}, \Sigma) \text{ and } B \cong C_0(\mathcal{W}^{(0)}).$$

Note:

- Topologically principal means $\{u \in \mathcal{W}^{(0)} \mid \mathcal{W}_u^u = \{u\}\}$ is dense in $\mathcal{W}^{(0)}$. Nontrivial groups are not top. principal.
- A twist is a central groupoid extension $\mathcal{W}^{(0)} \times \mathbb{T} \rightarrowtail \Sigma \twoheadrightarrow \mathcal{W}$. All cocycles induce a twist, but not all twists arise from a cocycle.
- $\mathcal{W}$ is called the **Weyl groupoid** and Σ is called the **Weyl twist** of the Cartan pair (A, B).

Back to the big picture

Question: Can $\mathcal{W}$ and Σ be described in terms of $\mathcal{S}, \mathcal{G}$, and c?

Our Weyl groupoid and Weyl twist

Suppose $\mathcal{S}, \mathcal{G}$, and c are as in Theorem 1, so $C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $C_r^*(\mathcal{G}, c)$. Let $\widehat{\mathcal{S}}$ denote the Gelfand dual of $C_r^*(\mathcal{S}, c)$, i.e. $C_r^*(\mathcal{S}, c) \cong C_0(\widehat{\mathcal{S}})$.

Theorem 2 (Duwenig-Gillaspy-N. 2021)

If there exists a continuous section of the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{S}$, then

$$C_r^*(\mathcal{G}, c) \cong C_r^*(\mathcal{G}/\mathcal{S} \ltimes \widehat{\mathcal{S}}, \sigma),$$

where σ is an explicitly defined continuous cocycle on $\mathcal{G}/\mathcal{S} \ltimes \widehat{\mathcal{S}}$.

Answer to a similar question

Theorem 2 (Duwenig-Gillaspy-N. 2021)

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Theorem (Ionescu-Kumjian-Renault-Sims-Williams 2021)

Let $\mathcal{G}$ be a second countable, locally compact Hausdorff (not necessarily étale) groupoid with a Haar system, and let $\mathcal{S}$ be a closed normal bundle of abelian groups with a Haar system. If $\mathcal{G}/\mathcal{S}$ is étale and topologically principal, then $C_r^*(\mathcal{S})$ is a Cartan subalgebra of $C_r^*(\mathcal{G})$ and

$$C_r^*(\mathcal{G}) \cong C_r^*(\widehat{\mathcal{S}} \rtimes \mathcal{G}/\mathcal{S}, \Sigma),$$

where Σ is a twist over $\widehat{\mathcal{S}} \rtimes \mathcal{G}/\mathcal{S}$.

Theorem 2 in action

Recall: $(\mathbb{Z}^5, *, \mathbf{c})$

Theorem 1: $C_r^*(\mathcal{S}_i)$ is Cartan in $C_r^*(\mathcal{G}, c)$ for
$\mathcal{S}_0 = \mathbb{Z} \times \mathbb{Z} \times \{0\} \times \{0\} \times \mathbb{Z}$
$\mathcal{S}_1 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times 2\mathbb{Z} \times \{0\}$
$\mathcal{S}_2 = 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \{0\}$

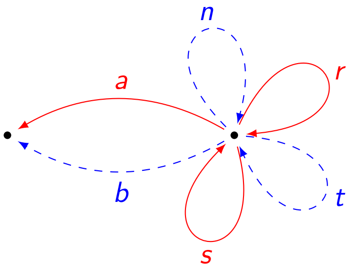
Theorem 2: The Weyl groupoids are
$\mathcal{W}_0 \cong \mathbb{Z}^2 \ltimes \mathbb{T}^3$
$\mathcal{W}_1 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}) \ltimes \mathbb{T}^4$
$\mathcal{W}_2 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}) \ltimes \mathbb{T}^4$
and $C_r^*(\mathcal{S}_0) \cong C(\mathbb{T}^3) \ncong C(\mathbb{T}^4) \cong C_r^*(\mathcal{S}_1) \cong C_r^*(\mathcal{S}_2)$.

A detour into higher rank graphs

Definition

A **higher rank graph** (or k-graph) is a countable small category Λ equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorization property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

Example of a 2-graph:



$$rn = nr$$
$$st = ts$$
$$rt = ns$$
$$sn = tr$$
$$an = br$$
$$at = bs$$

More about 2-graphs

Let Λ be a row-finite 2-graph with no sources and c_Λ be a cubical cocycle on Λ.

Let Λ^∞ denote the **infinite path space** of Λ.

Given Λ and c_Λ, we can construct

- a groupoid

$$\mathcal{G}_\Lambda := \{(x, l-m, y) \mid x, y \in \Lambda^\infty, l, m \in \mathbb{N}^2, \sigma^l(x) = \sigma^m(y)\}$$

  - multiplication: $(x, p, y)(y, q, z) = (x, p+q, z)$
  - inverse: $(x, p, y)^{-1} = (y, -p, x)$
- a continuous cocycle c on $\mathcal{G}_\Lambda$ [Kumjian-Pask-Sims 2015, Lemma 6.3].

Application of Theorem 1 to higher rank graphs

Question: What exactly do the subgroupoids $\mathcal{S} \subseteq \mathcal{G}_\Lambda$ look like that satisfy Theorem 1?

Recall: $\mathcal{S} \subseteq \mathcal{G}_\Lambda$ satisfies Theorem 1 if

- $\mathcal{S}$ is maximal among abelian subgroupoids of $\mathrm{Iso}(\mathcal{G}_\Lambda)$ on which c is symmetric.

- $\mathcal{S}$ is open and closed.
- $\mathcal{S}$ is normal.
- $\mathcal{S}$ is immediately centralizing.

Theorem 1 for k-graphs

For $x \in \Lambda^\infty$, let

$$[x] := \{y \in \Lambda^\infty \mid \sigma^l(y) = \sigma^m(x) \text{ for some } l, m \in \mathbb{N}^2\}.$$

Theorem 1 for k-graphs (N.-Reznikoff-Wright)

Let Λ be a row-finite 2-graph with no sources and c_Λ be a categorical cocycle on Λ. Let c be the continuous cocycle on $\mathcal{G}_\Lambda$ constructed from c_Λ according to [KPS 2015, Lemma 6.3]. For each $[x]$, fix $r_x \in \mathbb{Q} \cup \{\infty\}$. Define

$$\mathcal{S} := \bigcup_{[x]} \bigcup_{y \in [x]} \{(y, (m_1, m_2), y) \in \mathcal{G}_\Lambda \mid \frac{m_1}{m_2} = r_x\}.$$

If $\mathcal{S}$ is open and closed, then $C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $C_r^*(\mathcal{G}_\Lambda, c)$.

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Thank you!