An invitation to mean dimension of a dynamical system and the radius of comparison of its crossed product

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A rough outline

- Introduction.
- Free and minimal actions.
- Covering dimension.
- Mean dimension.
- Crossed products (very brief).
- Radius of comparison.
- Some results.

Introduction 1

Statement of the conjecture (explanations etc. afterwards):

Conjecture

Let G be a countable amenable group, let X be a compact metrizable space, and let T be a free minimal action of G on X. Then

$$\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T).$$

Here (much more detail below):

- $C^*(G, X, T)$ is the transformation group C*-algebra (crossed product).
- For any unital C*-algebra A, $\mathrm{rc}(A)$ is its radius of comparison.
- For any action T as in the conjecture (except not necessarily free or minimal), $\mathrm{mdim}(T)$ is its mean dimension.

Introduction 2

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

- For any unital C*-algebra A, rc(A) is its radius of comparison.
- For any action T of G on X, mdim(T) is its mean dimension.

rc(A) was introduced by Toms for the purpose of distinguishing counterexamples to an earlier form of the Elliott classification conjecture, before Z-stability was added to its hypotheses. Toms' algebras have no connection with dynamics.

mdim(T) was defined by Lindenstrauss and Weiss (following a suggestion of Gromov) for dynamical reasons having no connections with operator algebras. Its first application was the existence of a minimal homeomorphism T of a compact metrizable space X such that (X, T) does not embed in $([0,1]^{\mathbb{Z}}, \mathrm{shift})$.

Free and minimal 1

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

In all examples, etc., $G = \mathbb{Z}$, so the action comes from a homeomorphism.

Minimal: every orbit is dense. Equivalently: no nontrivial closed invariant sets.

- Example: Irrational rotations.
- Example: The trivial action on a one point space.
- Nonexample: Rational rotations.
- Nonexample: The shift on $[0,1]^{\mathbb{Z}}$. It has fixed points (constant sequences) whose orbits are obviously not dense, although it also has points with dense orbits.

Free and minimal 2

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

Minimal: every orbit is dense. Examples: irrational rotations and the trivial action on a point. Nonexamples: rational rotations and shifts.

Free: no $g \in G \setminus \{1\}$ has fixed points. That is, $gx = x$ implies $g = 1$.

- Example: Irrational rotations.
- Example: $+1$ in the p-adic integers.
- Nonexample: the trivial action on a one point space, unless $G = \{1\}$.
- Nonexample: The shift on $[0,1]^{\mathbb{Z}}$. Constant sequences are fixed points, and periodic sequences are periodic points.

The correct condition is surely "essentially free": definition omitted.

One reason for significance: since G is amenable, $C^*(G, X, T)$ is simple if and only if T is minimal and essentially free. (Archbold and Spielberg, 1994; earlier by Kawamura and Tomiyama 1990, but for nonamenable groups their theorem is not correct as stated: see Math. Reviews.)

Covering dimension 1: Open covers; order; refinement

We describe covering dimension, the basis of mean (covering) dimension.

Throughout, X is a compact metric space, and $\mathcal{U}$ and $\mathcal{V}$ are finite open covers of X. (In some proofs the covering requirement must be weakened: not done here.)

Definition

Let X be a compact metric space, and let $\mathcal{U}$ be a finite open cover of X. The *order* ord($\mathcal{U}$) of $\mathcal{U}$ is the least number $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n + 2$ distinct elements of $\mathcal{U}$ is empty.

For example, a cover by disjoint open sets (as is expected in a zero dimensional space) has order zero.

Definition

Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X. Then $\mathcal{V}$ *refines* $\mathcal{U}$ (written $\mathcal{V} \prec \mathcal{U}$) if every set in $\mathcal{V}$ is contained in some set in $\mathcal{U}$.

Covering dimension 2: $\mathcal{D}(\mathcal{U})$; dimension

$\mathrm{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty.

$\mathcal{V}$ refines $\mathcal{U}$ if $V \in \mathcal{V}$ implies there is $U \in \mathcal{U}$ such $V \subset U$.

Definition

Let X be a compact metric space, and let $\mathcal{U}$ be a finite open cover of X. Then $\mathcal{D}(\mathcal{U})$ is the least possible order of any refinement of $\mathcal{U}$.

Definition

Let X be a compact metric space. Then $\dim(X)$ (covering dimension) is the supremum of $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$ over all finite open covers $\mathcal{U}$ of X.

$\dim(X) \leqslant d$ says X has arbitrarily fine open covers $\mathcal{U}$ with $\mathrm{ord}(\mathcal{U}) \leqslant d$.

The Cantor set has dimension 0. $\dim([0,1]) \leqslant 1$: cover by short slightly overlapping intervals. $\dim([0,1]^2) \leqslant 2$: cover by small slightly overlapping hexagons. (Squares don't work.) Generally $\dim([0,1]^d) = d$. The lower bound uses the Brouwer Fixed Point Theorem.

Mean dimension 1: Heuristics

$\mathrm{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty.

$\dim(X) \leqslant d$ means X has arbitrarily fine open covers $\mathcal{U}$ with $\mathrm{ord}(\mathcal{U}) \leqslant d$.

$\dim([0,1]^d) = d$. More generally, if X is a finite complex, then $\dim(X)$ is the largest of the (naive) dimensions of its cells.

Mean dimension is supposed to be designed to make the mean dimension of the shift T on $([0,1]^d)^{\mathbb{Z}}$ (an infinite dimensional space) equal to d. Generally, $\mathrm{mdim}(T)$ is supposed to tell you "how much more of the dimension of X do you see with every iteration of T".

Mean dimension 2: Joins and images

$\mathrm{mdim}(T)$ is supposed to tell you "how much more of the dimension of X do you see with every iteration of T".

For the shift on $([0,1]^d)^{\mathbb{Z}}$, the answer is supposed to be d.

Definition

Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X. Then the *join* $\mathcal{U} \vee \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}.$$

This is the coarsest common refinement (up to equivalence).

Definition

Let X be a compact metric space, let $\mathcal{U}$ be a finite open cover of X, and let $T \colon X \to X$ be a homeomorphism. We define

$$T^{-1}(\mathcal{U}) = \{T^{-1}(U) \colon U \in \mathcal{U}\}.$$

Given $\mathcal{U}$, with n iterations of T you see: $\mathcal{U} \vee T^{-1}(\mathcal{U}) \vee \cdots \vee T^{-n+1}(\mathcal{U})$.

Mean dimension 3: Definition

Recall: $\mathrm{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty; $\mathcal{D}(\mathcal{U})$ is the least order of any refinement of $\mathcal{U}$; $\mathcal{U} \vee \mathcal{V} = \{U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$; and $T^{-1}(\mathcal{U})$ is taken setwise.

Definition

Let X be a compact metric space and let $T \colon X \to X$ be a homeomorphism. Denote by $\mathrm{Cov}(X)$ the set of finite open covers of X. Then the *mean dimension* of T is

$$\mathrm{mdim}(T) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee T^{-1}(\mathcal{U}) \vee \cdots \vee T^{-n+1}(\mathcal{U})\big)}{n}.$$

(One needs to check that the limit exists.)

Note: if $\dim(X) < \infty$ then $\mathrm{mdim}(T) = 0$.

For amenable groups, replace intervals in $\mathbb{Z}$ with Følner sets in the group.

Mean dimension 4: The shift

$$\mathrm{mdim}(T) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee T^{-1}(\mathcal{U}) \vee \cdots \vee T^{-n+1}(\mathcal{U})\big)}{n}.$$

The expression $\mathcal{D}\big(\mathcal{U} \vee T^{-1}(\mathcal{U}) \vee \cdots \vee T^{-n+1}(\mathcal{U})\big)$ tells you how much of the dimension of X one sees starting with $\mathcal{U}$ and applying T a total of n times. So we are looking at the "linear rate of growth of dimension with iteration of T".

If K is a finite complex, then the shift action on K^G does indeed have mean dimension equal to $\dim(K)$. (Lindenstrauss and Weiss for $K = [0,1]^d$, 2000.)

Crossed products

If G acts on a compact space X, via $(g, x) \mapsto gx$ then G also acts on $C(X)$, via $\alpha_g(f)(x) = f(g^{-1}x)$.

If G is discrete and $\alpha \colon G \to \mathrm{Aut}(A)$ is an action on a unital C*-algebra, then $C^*(G, A, \alpha)$ is generated by a copy of A and unitaries u_g for $g \in G$ which form a representation of G and implement the action:

$$u_g u_h = u_{gh}$$

for $g, h \in G$ and

$$u_g a u_g^* = \alpha_g(a)$$

for $g \in G$ and $a \in A$.

More formally (not quite the usual definition), $C^*(G, A, \alpha)$ is the universal C*-algebra generated in this way.

If $A = \mathbb{C}$ (that is, X at the top of the slide has one point), the action is trivial, and one gets the group C*-algebra $C^*(G)$.

Radius of comparison 1: Projections

Let A be a C*-algebra, and let $p, q \in A$ (or in $M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$) be projections. Recall that $p \precsim q$ if there is s (necessarily a partial isometry) such that $ss^* = p$ and $s^*s \leqslant q$.

Assume A is unital (and exact) and let $\mathrm{T}(A)$ be its tracial state space. Extend tracial states to $M_\infty(A)$ (without renormalizing). Blackadar's Second Fundamental Comparability Question asks whether, given projections $p, q \in M_\infty(A)$, if $\tau(p) < \tau(q)$ for all $\tau \in \mathrm{T}(A)$, it follows that $p \precsim q$. (Without exactness, one needs quasitraces.) Usually one should ask about simple C*-algebras, but even there counterexamples are known.

If A has few or no nontrivial projections, Blackadar's Second Fundamental Comparability Question doesn't tell us much. One must use positive elements instead. As preparation, rewrite: $p \precsim q$ if and only if there is v such that $vqv^* = p$. (Take s above to be vq.)

Radius of comparison 2: Cuntz comparison

If p and q are projections, then $p \precsim q$ if and only if there is v such that $vqv^* = p$.

Definition

Let A be a C*-algebra. For $a, b \in M_\infty(A)_+$, we say that a is *Cuntz subequivalent to b in A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $M_\infty(A)$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

It is easy to check that this gives the same relation on projections.

If $a, b \in (M_n)_+$, then $a \precsim_A b$ if and only if $\mathrm{rank}(a) \leqslant \mathrm{rank}(b)$.

If $f, g \in C(X)_+$, then $f \precsim_A g$ if and only if one has containment of the "open supports":

$$\{x \in X \colon f(x) \neq 0\} \subset \{x \in X \colon g(x) \neq 0\}.$$

This is not true without the sequence (v_n) and the limit in the definition.

Radius of comparison 3: $d_\tau(a)$ and $\mathrm{rc}(A)$

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

Definition

Let A be a unital C*-algebra, and let $\tau \in \mathrm{T}(A)$. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$ for $a \in M_\infty(A)_+$.

If a is a projection, then $d_\tau(a) = \tau(a)$. In general, it is the trace of the "open support of a". For example, if $f \in C(X)_+$ and τ comes from a probability measure μ on X, then

$$d_\tau(f) = \mu(\{x \in X \colon f(x) \neq 0\}).$$

Definition

Let A be a unital exact C*-algebra. The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is the infimum of all $r > 0$ such that whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim_A b$ ("A has r-comparison").

Radius of comparison 4: Motivation

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$, the trace of the "open support of a".

$\mathrm{rc}(A)$ is the infimum of all $r > 0$ such that A has r-comparison: whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim_A b$.

The quantity $\mathrm{rc}(C(X))$ is controlled by cancellation properties of vector bundles (projections) on closed subsets, and is about half the covering dimension $\dim(X)$. (Upper bound, probably Toms, 2006; lower bound, in a paper in preparation.) Note: $A = C([0,1]^N)$ has comparison of projections: even $\tau(p) \leqslant \tau(q)$ for all $\tau \in \mathrm{T}(A)$ implies $p \precsim_A q$, since all projections are trivial, but $\mathrm{rc}(C([0,1]^N))$ is about $N/2$.

The case $\mathrm{rc}(A) = 0$ is called *strict comparison*. Restricted to projections, it is Blackadar's Second Fundamental Comparability Question. One part of the Elliott classification conjecture says that for simple separable nuclear unital A satisfying the Universal Coefficient Theorem, this condition should be equivalent to classifiability.

Results 1

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

Theorem (Niu, preprint 2019)

If $G = \mathbb{Z}^d$, then $\mathrm{rc}(C^*(G, X, T)) \leqslant \frac{1}{2}\mathrm{mdim}(T)$.

There were earlier results, and Niu proves the conjecture for other groups under additional conditions on the action.

This inequality is false without some hypotheses. Example: the trivial action T of $\mathbb{Z}^d$ on a one point space X has $\mathrm{mdim}(T) = 0$, but $C^*(\mathbb{Z}, X, T) \cong C((S^1)^d)$ has radius of comparison at least $\frac{1}{2}d - 2$.

Results 2

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

There is one known construction of minimal homeomorphisms with nonzero mean dimension. After earlier work starting with Giol and Kerr (2010), the most general form is due to Dou (2017). He starts with a finite complex K, any countable amenable group G, and a "density parameter" $\rho \in (0, 1)$. Using a subset of G with density ρ, he constructs a minimal shift invariant subspace of K^G such that the restriction T of the shift to this subspace has mean dimension $\rho \dim(K)$.

Theorem (with Hirshberg, preprint 2020)

Suppose $k = \dim(K)$ is even and $H^k(K; \mathbb{Q}) \neq 0$. Let T be the subshift in Dou's construction. Then $\mathrm{rc}(C^*(G, X, T)) \geqslant \frac{1}{2}\mathrm{mdim}(T) - 1$.

Results 3

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

Dou's construction uses a finite complex K, a countable amenable group G, and a "density parameter" $\rho \in (0, 1)$, and gives a minimal subspace of K^G such that the restriction T of the shift to this subspace has mean dimension $\rho \dim(K)$.

Theorem (with Hirshberg, preprint 2020)

Suppose $k = \dim(K)$ is even and $H^k(K; \mathbb{Q}) \neq 0$. Let T be the subshift in Dou's construction. Then $\mathrm{rc}(C^*(G, X, T)) \geqslant \frac{1}{2}\mathrm{mdim}(T) - 1$.

We get a less good bound for general K and $\rho > \frac{1}{2}$, which gets worse as ρ decreases.

The methods work for arbitrary countable amenable G, and don't require minimality.

Results 4: Heuristics 1

Conjecture

Let T be a free minimal action of a countable amenable group G on a compact metrizable space X. Then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

Recall that $C^*(G, X, T)$ is generated by a copy of $C(X)$ and a unitary representation $g \mapsto u_g$ of G, such that $(u_g f u_g^*)(x) = f(g^{-1}x)$.

Take $G = \mathbb{Z}$, so $T \colon X \to X$ is a homeomorphism. Suppose $Y \subset X$ is closed and has nonempty interior, and $Y, T^{-1}(Y), \ldots, T^{-n+1}(Y)$ are disjoint. Then one sees inside the crossed product something that looks like (up to discrepancies on ∂Y) the algebra $M_n(C(Y))$, with f seen as

$$\mathrm{diag}\big(f|_Y, f|_{T^{-1}(Y)}, \ldots, f|_{T^{-n+1}(Y)}\big)$$

or

$$\mathrm{diag}\big(f|_Y, f \circ T^{-1}|_Y, \ldots, f \circ T^{-n+1}|_Y\big),$$

and with the generating unitary u contributing a shift.

This suggests Rokhlin towers. Niu's $\mathrm{rc}(C^*(\mathbb{Z}^d, X, T)) \leqslant \frac{1}{2}\mathrm{mdim}(T)$, as well as previous work on upper bounds, uses Rokhlin tower methods.

Results 5: Heuristics 2

Theorem (with Hirshberg, preprint 2020)

Suppose $k = \dim(K)$ is even and $H^k(K; \mathbb{Q}) \neq 0$. Let T be the subshift in Dou's construction. Then $\mathrm{rc}(C^*(G, X, T)) \geqslant \frac{1}{2}\mathrm{mdim}(T) - 1$.

In $C^*(G, X, T)$, we sort of see $M_n(C(Y))$, with f appearing as

$$\mathrm{diag}\big(f|_Y, f \circ T^{-1}|_Y, \ldots, f \circ T^{-n+1}|_Y\big).$$

As a model, just consider

$$f \mapsto \mathrm{diag}\big(f, f \circ T^{-1}, \ldots, f \circ T^{-n+1}\big) \in M_n(C(X)).$$

Projections in $M_\infty(C(X))$ correspond to vector bundles over X. If E is not a subbundle of a higher rank vector bundle F, and this can be detected by Chern classes in $H^k(K; \mathbb{Z})$, this operation gives

$$V = E \oplus T^*(E) \oplus \cdots \oplus (T^{n-1})^*(E) \quad \text{and} \quad W = F \oplus \cdots \oplus (T^{n-1})^*(F).$$

Here $\mathrm{rank}(W) - \mathrm{rank}(V) = n[\mathrm{rank}(F) - \mathrm{rank}(E)]$, and V is not a subbundle of W; one uses Chern classes and cup products in cohomology.