C^*-envelopes of tensor algebras of product systems

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Tensor algebras of correspondences

Let A be a C^*-algebra. A *correspondence* $\mathcal{E}\colon A \rightsquigarrow A$ is a right Hilbert A-module with a nondegenerate *-homomorphism $\varphi\colon A \to \mathbb{B}(\mathcal{E})$.

The Fock space of $\mathcal{E}$ is the Hilbert A-module $\mathcal{E}^+ := \bigoplus_{n=0}^{\infty} \mathcal{E}^{\otimes n}$.

For $\xi \in \mathcal{E}$, let $t_\xi := \xi\otimes$ be the 'creation' operator on $\mathcal{E}^+$.

Let $t_\infty\colon \mathcal{E} \to \mathbb{B}(\mathcal{E}^+)$, $\xi \mapsto t_\xi$ and let $\varphi_\infty\colon A \to \mathbb{B}(\mathcal{E}^+)$ be the diagonal left action.

The Toeplitz C^*-algebra $\mathcal{T}_\mathcal{E}$ of $\mathcal{E}$ is the C^*-algebra generated by $\varphi_\infty(A) + t_\infty(\mathcal{E})$. The tensor algebra $\mathcal{T}^+(\mathcal{E})$ of $\mathcal{E}$ is the closed subalgebra of $\mathcal{T}_\mathcal{E}$ generated by $\varphi_\infty(A) + t_\infty(\mathcal{E})$.

Cuntz–Pimsner algebras

The gauge action: $\mathcal{T}_\mathcal{E}$ carries a continuous action $\bar{\gamma}$ of $\mathbb{T}$ such that

$$\bar{\gamma}_z(\varphi_\infty(a)) = \varphi_\infty(a) \quad \text{and} \quad \bar{\gamma}_z(t_\xi) = z\, t_\xi.$$

Thus $\mathcal{T}_\mathcal{E} = \overline{\bigoplus}_{n\in\mathbb{Z}} \mathcal{T}_\mathcal{E}^n$ with $\mathcal{T}_\mathcal{E}^m \cdot \mathcal{T}_\mathcal{E}^n \subset \mathcal{T}_\mathcal{E}^{m+n}$ and $(\mathcal{T}_\mathcal{E}^n)^* = \mathcal{T}_\mathcal{E}^{-n}$; there is a faithful conditional expectation $E^{\bar{\gamma}} \colon \mathcal{T}_\mathcal{E} \to \mathcal{T}_\mathcal{E}^0$ vanishing on $\mathcal{T}_\mathcal{E}^n$ for $n \neq 0$.

- The Cuntz–Pimsner algebra $\mathcal{O}_\mathcal{E}$ of $\mathcal{E}$ is the quotient $\mathcal{T}_\mathcal{E}/\mathbb{K}(\mathcal{E}^+ I_\mathcal{E})$, where $I_\mathcal{E} := \varphi^{-1}(\mathbb{K}(\mathcal{E})) \cap (\ker\varphi)^\perp \triangleleft A$.
- The gauge action on $\mathcal{T}_\mathcal{E}$ descends to a gauge action $\gamma \colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_\mathcal{E})$;

An important property of $\mathcal{O}_\mathcal{E}$: a *-homomorphism $\hat{\pi} \colon \mathcal{O}_\mathcal{E} \to B$ is faithful on the fixed-point algebra $\mathcal{O}_\mathcal{E}^0$ for γ iff it is faithful on A.

◇ $\mathcal{T}_\mathcal{E}$ and $\mathcal{O}_\mathcal{E}$ can also be characterised via universal properties.

C^*-envelopes

An operator algebra C on a Hilbert space $\mathcal{H}$ is a closed subalgebra of $\mathbb{B}(\mathcal{H})$.

(Arveson, 69): Let C be an operator algebra contained in a C^*-algebra B with $\mathrm{C}^*(C) = B$. An ideal J in B is a boundary ideal for C if the quotient map $B \to B/J$ is completely isometric on C. A boundary ideal J of B is the Shilov boundary for C if it contains every other boundary ideal for C.

A C^*-cover of C is a pair (B, ρ), where B is a C^*-algebra and $\rho\colon C \to B$ is a completely isometric homomorphism such that $\mathrm{C}^*(\rho(C)) = B$.

The C^*-envelope of C is a C^*-cover $(\mathrm{C}^*_{\mathrm{env}}(C), \iota)$ satisfying the following:

- ◇ if (B, ρ) is a C^*-cover of C, then there exists a *-homomorphism $\pi\colon B \to \mathrm{C}^*_{\mathrm{env}}(C)$ such that $\pi \circ \rho = \iota$.

(Hamana, 79): the C^*-envelope of C exists and is unique up to isomorphism.

Cuntz–Pimsner algebras as C^*-envelopes of tensor algebras

If $\mathcal{E} = \mathbb{C} \colon \mathbb{C} \rightsquigarrow \mathbb{C}$, then $\mathcal{T}(\mathbb{C})^+$ can be identified with the algebra of analytic functions on the unit disc, and thus

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}(\mathbb{C})^+) = \mathrm{C}(\mathbb{T}) \cong \mathcal{O}_{\mathbb{C}}.$$

Theorem (Muhly–Solel, Fowler–Muhly–Raeburn, Katsoulis–Kribs)

Let $\mathcal{E} \colon A \rightsquigarrow A$ be a correspondence, and let $\mathcal{T}(\mathcal{E})^+$ be the tensor algebra of $\mathcal{E}$. Then the C^*-envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}(\mathcal{E})^+)$ is canonically isomorphic to the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$.

Consequence: the C^*-envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}(\mathcal{E})^+)$ automatically carries a continuous action γ of $\mathbb{T}$ for which the quotient map $\phi_\lambda \colon \mathcal{T}_{\mathcal{E}} \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_{\mathcal{E}}^+)$ is gauge-equivariant.

Product systems

Let P be a submonoid of a group G. A product system over P of A-correspondences consists of:

(i) a correspondence $\mathcal{E}_p \colon A \rightsquigarrow A$ for each $p \in P$, where $\mathcal{E}_e = A$ is the identity correspondence;

(ii) associative correspondence isomorphisms $\mu_{p,q} \colon \mathcal{E}_p \otimes_A \mathcal{E}_q \xrightarrow{\cong} \mathcal{E}_{pq}$, also called *multiplication maps*, for all $p, q \in P$, where $\mu_{e,p}$ and $\mu_{p,e}$ implement the left and right actions of A on $\mathcal{E}_p$, respectively.

- ◇ **Examples**: the canonical product system $\mathbb{C}^P$ of one-dimensional fibres; from semigroups of endomorphisms on unital C^*-algebras etc.
- ◇ If $\mathcal{E}_1 \colon A \rightsquigarrow A$ is a correspondence, then setting $\mathcal{E}_n := \mathcal{E}^{\otimes n}$, we have that $\mathcal{E} = (\mathcal{E}_n)_{n \in \mathbb{N}}$ is a product system over $\mathbb{N}$.

The Fock representation

A representation of a product system $\mathcal{E} = (\mathcal{E}_p)_{p\in P}$ in a C^*-algebra B consists of linear maps $\pi_p\colon \mathcal{E}_p \to B$, for all $p \in P \setminus \{e\}$, and a *-homomorphism $\pi_e\colon A \to B$, satisfying the following two axioms:

(i) $\pi_p(\xi)\pi_q(\eta) = \pi_{pq}(\xi\eta)$ for all $p, q \in P$, $\xi \in \mathcal{E}_p$ and $\eta \in \mathcal{E}_q$;

(ii) $\pi_p(\xi)^*\pi_p(\eta) = \pi_e(\langle \xi \mid \eta \rangle)$, for all $p \in P$ and $\xi, \eta \in \mathcal{E}_p$.

The Fock space of $\mathcal{E}$ is $\mathcal{E}^+ \coloneqq \bigoplus_{p\in P} \mathcal{E}_p$. There is a canonical injective representation $\psi = \{\psi_p\}_{p\in P}$ of $\mathcal{E}$ in $\mathbb{B}(\mathcal{E}^+)$ called the Fock representation.

Definition

The Fock algebra $\mathcal{T}_\lambda(\mathcal{E})$ of $\mathcal{E}$ is the C^*-algebra generated by the range of ψ. The tensor algebra $\mathcal{T}_\lambda(\mathcal{E})^+$ of $\mathcal{E}$ is the closed subalgebra of $\mathcal{T}_\lambda(\mathcal{E})$ generated by the range of ψ. That is,

$$\mathcal{T}_\lambda(\mathcal{E})^+ = \overline{\mathrm{span}}\{\psi_p(\xi) \mid p \in P, \xi \in \mathcal{E}_p\}.$$

Strongly covariant representations

Fowler's C^*-algebra $\mathrm{C}^*_{\mathrm{rep}}(\mathcal{E})$ is the C^*-algebra generated by a universal representation $\tilde{t} = \{\tilde{t}_p\}_{p\in P}$ of $\mathcal{E}$. There is a *coaction*

$$\tilde{\delta}\colon \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E}) \to \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E}) \otimes \mathrm{C}^*(G), \quad \tilde{t}_p(\xi) \mapsto \tilde{t}_p(\xi) \otimes u_p.$$

Thus $\mathrm{C}^*_{\mathrm{rep}}(\mathcal{E}) = \overline{\bigoplus} \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E})_g$ is *topologically G-graded*.
For each $F \subset G$ finite, we associate a closed submodule $\mathcal{E}_F$ of $\mathcal{E}^+$, and a *-homomorphism $t_F\colon \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E})_e \to \mathbb{B}(\mathcal{E}_F)$ such that:

- If $F_1 \subset F_2$ are finite subsets of G, then $\mathcal{E}_{F_2} \subset \mathcal{E}_{F_1}$.
- Setting $\|b\|_F := \|t_F(b)\|$, we have $\|b\|_{F_2} \leq \|b\|_{F_1}$. We define an ideal $J_e \lhd \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E})_e$ by $J_e := \left\{ b \in \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E})_e \,\middle|\, \lim_F \|b\|_F = 0 \right\}$.

Definition (S., 19)

A representation $\pi = \{\pi_p\}_{p\in P}$ of $\mathcal{E}$ in a C^*-algebra B is strongly covariant if the induced *-homomorphism $\tilde{\pi}\colon \mathrm{C}^*_{\mathrm{rep}}(\mathcal{E}) \to B$ vanishes on J_e.

Covariance algebra

The covariance algebra $A \times_{\mathcal{E}} P$ of $\mathcal{E}$ is the quotient $\mathrm{C}^*_{\text{rep}}(\mathcal{E})/\langle J_e \rangle$. Let $j = \{j_p\}_{p \in P}$ be induced representation of $\mathcal{E}$ in $A \times_{\mathcal{E}} P$. The coaction on $\mathrm{C}^*_{\text{rep}}(\mathcal{E})$ induces a coaction $\delta \colon A \times_{\mathcal{E}} P \to (A \times_{\mathcal{E}} P) \otimes \mathrm{C}^*(G)$.

Theorem (S., 19)

The representation $j = \{j_p\}_{p \in P}$ is injective and a *-homomorphism $\hat{\pi} \colon A \times_{\mathcal{E}} P \to B$ is faithful on the fixed-point algebra $(A \times_{\mathcal{E}} P)^{\delta}$ if and only if it is faithful on A.

Examples:

- ◇ The full analogue of the Toeplitz algebra $\mathcal{T}_\lambda(P)$ of P.
- ◇ If $\mathcal{E} = \mathbb{C}^P$, then $\mathbb{C} \times_{\mathbb{C}^P} P$ is canonically isomorphic to the full analogue of the boundary quotient $\partial\mathcal{T}_\lambda(P)$ **(Laca-S., 21)**.

C^*-envelopes of cosystems

- If $\mathcal{E}$ is a compactly aligned product system over an abelian lattice order, then $A \times_{\mathcal{E}} P \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$ canonically **(DorOn-Katsoulis, 20)**.

(DKKLL, 21): In general, there is a gauge coaction $\overline{\delta}\colon \mathcal{T}_\lambda(\mathcal{E}) \to \mathcal{T}_\lambda(\mathcal{E}) \otimes \mathrm{C}^*(G)$, which restricts to a coaction $\overline{\delta}^+$ on $\mathcal{T}_\lambda(\mathcal{E})^+$.

The triple $(\mathcal{T}_\lambda(\mathcal{E})^+, G, \overline{\delta}^+)$ is a *cosystem*, and we can consider the coaction C^*-envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+, G, \overline{\delta}^+)$.

Theorem (DorOn-Kakariadis-Katsoulis-Laca-Li, 20)

Let P be a right LCM submonoid of a group G and let $\mathcal{E}$ be a compactly aligned product system over P. Then there is a canonical isomorphism

$$\mathrm{C}^*_r\left(([A \times_{\mathcal{E}} P]_g)_{g\in G}\right) \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+, G, \overline{\delta}^+).$$

(KKLL, 21): $\partial\mathcal{T}_\lambda(P) \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(P)^+, G, \overline{\delta}^+)$ canonically.

Reduced covariance algebras as C^*-envelopes

Theorem (S., 21)

Let P be a submonoid of a group G and let $\mathcal{E} = (\mathcal{E}_p)_{p\in P}$ be a product system over P with coefficient C^*-algebra A. The following C^*-algebras associated to $\mathcal{E}$ are canonically isomorphic:

(1) *the C^*-envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$;*

(2) *the reduced cross sectional C^*-algebra of the Fell bundle $([A \times_{\mathcal{E}} P]_g)_{g\in G}$ associated to the covariance algebra of $\mathcal{E}$;*

(3) *the C^*-envelope of the cosystem $(\mathcal{T}_\lambda(\mathcal{E})^+, G, \overline{\delta}^+)$.*

Note that $A \times_{\mathcal{E}} P \cong \mathrm{C}^*\left(([A \times_{\mathcal{E}} P]_g)_{g\in G}\right)$.

Reduced covariance algebra and coaction C^*-envelope

Proposition (S., 21)

Let $\hat{\pi}\colon \mathcal{T}_\lambda(\mathcal{E}) \to B$ be a *-homomorphism induced by an injective representation $\pi = \{\pi_p\}_{p\in P}$ of $\mathcal{E}$ in B. Suppose that there exists a conditional expectation $E_\pi\colon \hat{\pi}(\mathcal{T}_\lambda(\mathcal{E})) \to \hat{\pi}(\mathcal{T}_\lambda(\mathcal{E})_e)$ such that the diagram

$$\begin{array}{ccc} \mathcal{T}_\lambda(\mathcal{E}) & \xrightarrow{\hat{\pi}} & \hat{\pi}(\mathcal{T}_\lambda(\mathcal{E})) \\ \downarrow E_\lambda & & \downarrow E_\pi \\ \mathcal{T}_\lambda(\mathcal{E})_e & \xrightarrow{\hat{\pi}} & \hat{\pi}(\mathcal{T}_\lambda(\mathcal{E})_e) \end{array}$$

commutes. Then the restriction of $\hat{\pi}$ to the tensor algebra $\mathcal{T}_\lambda(\mathcal{E})^+$ is completely isometric.

This implies $\mathrm{C}^*_r(([A \times_{\mathcal{E}} P]_g)_{g\in G}) \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+, G, \bar{\delta}^+)$ ((2) $\cong$ (3)).

Idea of proof

Consider $b := \sum_s \psi_s(\xi_s) \in \mathcal{T}_\lambda(\mathcal{E})^+$ and $\eta = \sum_r \eta_r \in \mathcal{E}^+ = \bigoplus_{p\in P} \mathcal{E}_p$.
Setting $\eta'_p := \sum_{sr=p} \psi_s(\xi_s)\eta_r$ and using that π is injective, we have that

$$\|\sum_p \pi_e(\langle \eta'_p \,|\, \eta'_p\rangle)\| = \|\sum_p \langle \eta'_p \,|\, \eta'_p\rangle\| = \|\sum_s \psi(\xi_s)\eta\|^2 = \|b\eta\|^2. \quad (\dagger)$$

Now in B

$$\begin{aligned}\sum_{p,q} \pi_p(\eta'_p)^*\pi_q(\eta'_q) &= \big(\sum_r \pi_r(\eta_r)\big)^*\big(\sum_{s,t} \pi_s(\xi_s)^*\pi_t(\xi_t)\big)\big(\sum_r \pi_r(\eta_r)\big) \\ &\leq \|\sum_s \pi_s(\xi_s)\|^2 \sum_{r,r'} \pi_r(\eta_r)^*\pi_{r'}(\eta_{r'}).\end{aligned} \quad (\dagger\dagger)$$

Applying E_π yields $\|\sum_s \psi_s(\xi_s)\eta\|^2 \leq \|\sum_s \pi_s(\xi_s)\|^2\|\eta\|^2 = \|\hat{\pi}(b)\|^2\|\eta\|^2$.
This shows that $\hat{\pi}$ is isometric on $\mathcal{T}_\lambda(\mathcal{E})^+$.

An inequality for completely isometric homomorphisms

Lemma (S., 21)

Let $\hat{\pi}\colon \mathcal{T}_\lambda(\mathcal{E})^+ \to B$ *be a completely isometric homomorphism induced by a representation* $\pi = \{\pi_p\}_{p\in P}$ *of* $\mathcal{E}$ *in* B*. Then for every* $n \geq 1$*, finite sets* $F_1, F_2, \ldots, F_n \subset P$ *and choice of elements* $\xi_p \in \mathcal{E}_p$ *for* $p \in F_i$*,* $i = 1, \ldots, n$*, we have*

$$\|\sum_{i=1}^{n} \sum_{p\in F_i} \pi_e(\langle \xi_p \,|\, \xi_p \rangle)\| \leq \|\sum_{i=1}^{n} \sum_{p,q\in F_i} \pi_p(\xi_p)^* \pi_q(\xi_q)\|.$$

Idea: There is $b \in \mathbb{M}_n(\mathcal{T}_\lambda(\mathcal{E})^+)$ such that

$$\|\sum_{i=1}^{n} \sum_{p,q\in F_i} \psi_p(\xi_p)^* \psi_q(\xi_q)\| = \|b^* b\| = \|b\|^2.$$

Conditional expectation on the C^*-envelope of $\mathcal{T}_\lambda(\mathcal{E})^+$

Proposition (S., 21)

Let $\pi = \{\pi_p\}_{p\in P}$ be an injective strongly covariant representation of $\mathcal{E}$ in a C^*-algebra B and let $\hat{\pi}\colon A \times_{\mathcal{E}} P \to B$ be the induced *-homomorphism. Then the following are equivalent:

(1) *There exists a conditional expectation $E_\pi \colon \hat{\pi}(A \times_{\mathcal{E}} P) \to \hat{\pi}((A \times_{\mathcal{E}} P)^\delta)$ such that $\hat{\pi} \circ E^\delta = E_\pi \circ \hat{\pi}$.*

(2) *For every $n \geq 1$, finite sets $F_1, F_2, \ldots, F_n \subset P$ and elements $\xi_p \in \mathcal{E}_p$ for $p \in F_i$, $i = 1, \ldots, n$, we have that*

$$\|\sum_{i=1}^{n} \sum_{p\in F_i} \pi_e(\langle \xi_p \,|\, \xi_p \rangle)\| \leq \|\sum_{i=1}^{n} \sum_{p,q\in F_i} \pi_p(\xi_p)^* \pi_q(\xi_q)\|.$$

This implies $\mathrm{C}^*_r\,(([A \times_{\mathcal{E}} P]_g)_{g\in G}) \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$ ((1) $\cong$ (2)).

Co-universal property of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$

Corollary (S.,21)

Let $(\mathcal{T}_\lambda(\mathcal{E})_g)_{g\in G}$ be the Fell bundle associated to the coaction $\bar{\delta}$ of G on $\mathcal{T}_\lambda(\mathcal{E})$. Then the C^*-envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$ satisfies the following:

(1) *there is a coaction δ_Λ on $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$ for which the representation of $\mathcal{E}$ induced by the inclusion of $\mathcal{T}_\lambda(\mathcal{E})^+$ is gauge-compatible;*

(2) *if (B, G, γ) is a coaction and $\pi = \{\pi_p\}_{p\in P}$ is an injective representation of $\mathcal{E}$ in B that is gauge-compatible with γ and induces a surjective *-homomorphism $\hat{\pi}\colon \mathrm{C}^*((\mathcal{T}_\lambda(\mathcal{E})_g)_{g\in G}) \to B$, then there exists a $\gamma - \delta_\Lambda$-equivariant surjective *-homomorphism $\rho\colon B \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)$ that identifies $\mathcal{E}$.*

Thanks!